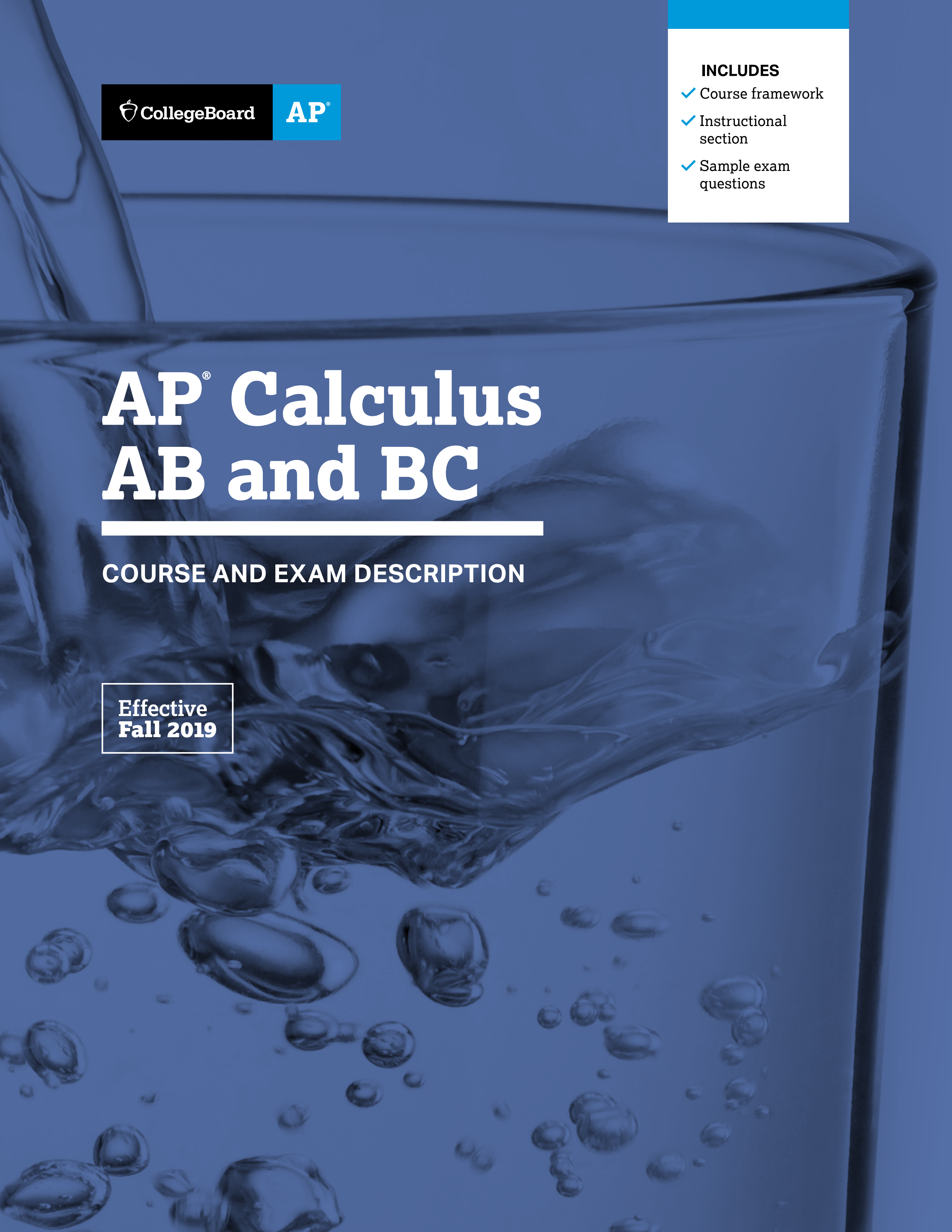

CollegeBoard AP

INCLUDES
- ✓ Course framework
- ✓ Instructional section
- ✓ Sample exam questions

AP® Calculus AB and BC

COURSE AND EXAM DESCRIPTION

Effective Fall 2019

AP® CollegeBoard

AP® Calculus AB and BC

COURSE AND EXAM DESCRIPTION

Effective
Fall 2019

AP COURSE AND EXAM DESCRIPTIONS ARE UPDATED PERIODICALLY

Please visit AP Central (apcentral.collegeboard.org) to determine whether a more recent course and exam description is available.

About College Board

College Board is a mission-driven not-for-profit organization that connects students to college success and opportunity. Founded in 1900, College Board was created to expand access to higher education. Today, the membership association is made up of more than 6,000 of the world's leading educational institutions and is dedicated to promoting excellence and equity in education. Each year, College Board helps more than seven million students prepare for a successful transition to college through programs and services in college readiness and college success—including the SAT® and the Advanced Placement® Program. The organization also serves the education community through research and advocacy on behalf of students, educators, and schools.

For further information, visit **collegeboard.org**.

AP Equity and Access Policy

College Board strongly encourages educators to make equitable access a guiding principle for their AP programs by giving all willing and academically prepared students the opportunity to participate in AP. We encourage the elimination of barriers that restrict access to AP for students from ethnic, racial, and socioeconomic groups that have been traditionally underrepresented. Schools should make every effort to ensure their AP classes reflect the diversity of their student population. College Board also believes that all students should have access to academically challenging course work before they enroll in AP classes, which can prepare them for AP success. It is only through a commitment to equitable preparation and access that true equity and excellence can be achieved.

Designers: Sonny Mui and Bill Tully

Visit College Board on the Web: **collegeboard.org**.

Contents

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About AP

College Board's Advanced Placement® Program (AP®) enables willing and academically prepared students to pursue college-level studies—with the opportunity to earn college credit, advanced placement, or both—while still in high school. Through AP courses in 38 subjects, each culminating in a challenging exam, students learn to think critically, construct solid arguments, and see many sides of an issue—skills that prepare them for college and beyond. Taking AP courses demonstrates to college admission officers that students have sought the most challenging curriculum available to them, and research indicates that students who score a 3 or higher on an AP Exam typically experience greater academic success in college and are more likely to earn a college degree than non-AP students. Each AP teacher's syllabus is evaluated and approved by faculty from some of the nation's leading colleges and universities, and AP Exams are developed and scored by college faculty and experienced AP teachers. Most four-year colleges and universities in the United States grant credit, advanced placement, or both on the basis of successful AP Exam scores—more than 3,300 institutions worldwide annually receive AP scores.

AP Course Development

In an ongoing effort to maintain alignment with best practices in college-level learning, AP courses and exams emphasize challenging, research-based curricula aligned with higher education expectations.

Individual teachers are responsible for designing their own curriculum for AP courses, selecting appropriate college-level readings, assignments, and resources. This course and exam description presents the content and skills that are the focus of the corresponding college course and that appear on the AP Exam. It also organizes the content and skills into a series of units that represent a sequence found in widely adopted college textbooks and that many AP teachers have told us they follow in order to focus their instruction. The intention of this publication is to respect teachers' time and expertise by providing a roadmap that they can modify and adapt to their local priorities and preferences. Moreover, by organizing the AP course content and skills into units, the AP Program is able to provide teachers and students with free formative assessments—Personal Progress Checks—that teachers can assign throughout the year to measure student progress as they acquire content knowledge and develop skills.

Enrolling Students: Equity and Access

College Board strongly encourages educators to make equitable access a guiding principle for their AP programs by giving all willing and academically prepared students the opportunity to participate in AP. We encourage the elimination of barriers that restrict access to AP for students from ethnic, racial, and socioeconomic groups that have been traditionally underserved. College Board also believes that all students should have access to academically challenging coursework before they enroll in AP classes, which can prepare them for AP success. It is only through a commitment to equitable preparation and access that true equity and excellence can be achieved.

Offering AP Courses: The AP Course Audit

The AP Program unequivocally supports the principle that each school implements its own curriculum that will enable students to develop the content understandings and skills described in the course framework.

While the unit sequence represented in this publication is optional, the AP Program does have a short list of curricular and resource requirements that must be fulfilled before a school can label a course "Advanced Placement" or "AP." Schools wishing to offer AP courses must participate in the AP Course Audit, a process through which AP teachers' course materials are reviewed by college faculty. The AP Course Audit was created to provide teachers and administrators with clear guidelines on curricular and resource requirements for AP courses and to help colleges and universities validate courses marked "AP" on students' transcripts. This process ensures that AP teachers' courses meet or exceed the curricular and resource expectations that college and secondary school faculty have established for college-level courses.

The AP Course Audit form is submitted by the AP teacher and the school principal (or designated administrator) to confirm awareness and understanding of the curricular and resource requirements. A syllabus or course outline, detailing how course requirements are met, is submitted by the AP teacher for review by college faculty.

Please visit **collegeboard.org/apcourseaudit** for more information to support the preparation and submission of materials for the AP Course Audit.

How the AP Program Is Developed

The scope of content for an AP course and exam is derived from an analysis of hundreds of syllabi and course offerings of colleges and universities. Using this research and data, a committee of college faculty and expert AP teachers work within the scope of the corresponding college course to articulate what students should know and be able to do upon the completion of the AP course. The resulting course framework is the heart of this course and exam description and serves as a blueprint of the content and skills that can appear on an AP Exam.

The AP Test Development Committees are responsible for developing each AP Exam, ensuring the exam questions are aligned to the course framework. The AP Exam development process is a multiyear endeavor; all AP Exams undergo extensive review, revision, piloting, and analysis to ensure that questions are accurate, fair, and valid, and that there is an appropriate spread of difficulty across the questions.

Committee members are selected to represent a variety of perspectives and institutions (public and private, small and large schools and colleges), and a range of gender, racial/ethnic, and regional groups. A list of each subject's current AP Test Development Committee members is available on **apcentral.collegeboard.org**.

Throughout AP course and exam development, College Board gathers feedback from various stakeholders in both secondary schools and higher education institutions. This feedback is carefully considered to ensure that AP courses and exams are able to provide students with a college-level learning experience and the opportunity to demonstrate their qualifications for advanced placement or college credit.

How AP Exams Are Scored

The exam scoring process, like the course and exam development process, relies on the expertise of both AP teachers and college faculty. While multiple-choice questions are scored by machine, the free-response questions and through-course performance assessments, as applicable, are scored by thousands of college faculty and expert AP teachers. Most are scored at the annual AP Reading, while a small portion is scored online. All AP Readers are thoroughly trained, and their work is monitored throughout the Reading for fairness and consistency. In each subject, a highly respected college faculty member serves as Chief Faculty Consultant and, with the help of AP Readers in leadership positions, maintains the accuracy of the scoring standards. Scores on the free-response questions and performance assessments are weighted and combined with the results of the computer-scored multiple-choice questions, and this raw score is converted into a composite AP score on a 1–5 scale.

AP Exams are **not** norm-referenced or graded on a curve. Instead, they are criterion-referenced, which means that every student who meets the criteria for an AP score of 2, 3, 4, or 5 will receive that score, no matter how many students that is. The criteria for the number of points students must earn on the AP Exam to receive scores of 3, 4, or 5—the scores that research consistently validates for credit and placement purposes—include:

- The number of points successful college students earn when their professors administer AP Exam questions to them.
- The number of points researchers have found to be predictive that an AP student will succeed when placed into a subsequent higher-level college course.
- Achievement-level descriptions formulated by college faculty who review each AP Exam question.

Using and Interpreting AP Scores

The extensive work done by college faculty and AP teachers in the development of the course and exam and throughout the scoring process ensures that AP Exam scores accurately represent students' achievement in the equivalent college course. Frequent and regular research studies establish the validity of AP scores as follows:

AP Score	Credit Recommendation	College Grade Equivalent
5	Extremely well qualified	A
4	Well qualified	A–, B+, B
3	Qualified	B–, C+, C
2	Possibly qualified	n/a
1	No recommendation	n/a

While colleges and universities are responsible for setting their own credit and placement policies, most private colleges and universities award credit and/or advanced placement for AP scores of 3 or higher. Additionally, most states in the U.S. have adopted statewide credit policies that ensure college credit for scores of 3 or higher at public colleges and universities. To confirm a specific college's AP credit/placement policy, a search engine is available at **apstudent.collegeboard.org/creditandplacement/search-credit-policies**.

BECOMING AN AP READER

Each June, thousands of AP teachers and college faculty members from around the world gather for seven days in multiple locations to evaluate and score the free-response sections of the AP Exams. Ninety-eight percent of surveyed educators who took part in the AP Reading say it was a positive experience.

There are many reasons to consider becoming an AP Reader, including opportunities to:

- **Bring positive changes to the classroom:** Surveys show that the vast majority of returning AP Readers—both high school and college educators—make improvements to the way they teach or score because of their experience at the AP Reading.
- **Gain in-depth understanding of AP Exam and AP scoring standards:** AP Readers gain exposure to the quality and depth of the responses from the entire pool of AP Exam takers, and thus are better able to assess their students' work in the classroom.
- **Receive compensation:** AP Readers are compensated for their work during the Reading. Expenses, lodging, and meals are covered for Readers who travel.
- **Score from home:** AP Readers have online distributed scoring opportunities for certain subjects. Check **collegeboard.org/apreading** for details.
- **Earn Continuing Education Units (CEUs):** AP Readers earn professional development hours and CEUs that can be applied to PD requirements by states, districts, and schools.

How to Apply

Visit **collegeboard.org/apreading** for eligibility requirements and to start the application process.

AP Resources and Supports

By completing a simple activation process at the start of the school year, teachers and students receive access to a robust set of classroom resources.

AP Classroom

AP Classroom is a dedicated online platform designed to support teachers and students throughout their AP experience. The platform provides a variety of powerful resources and tools to provide yearlong support to teachers and enable students to receive meaningful feedback on their progress.

UNIT GUIDES

Appearing in this publication and on AP Classroom, these planning guides outline all required course content and skills, organized into commonly taught units. Each unit guide suggests a sequence and pacing of content, scaffolds skill instruction across units, organizes content into topics, and provides tips on taking the AP Exam.

PERSONAL PROGRESS CHECKS

Formative AP questions for every unit provide feedback to students on the areas where they need to focus. Available online, Personal Progress Checks measure knowledge and skills through multiple-choice questions with rationales to explain correct and incorrect answers, and free-response questions with scoring information. Because the Personal Progress Checks are formative, the results of these assessments cannot be used to evaluate teacher effectiveness or assign letter grades to students, and any such misuses are grounds for losing school authorization to offer AP courses.*

PROGRESS DASHBOARD

This dashboard allows teachers to review class and individual student progress throughout the year. Teachers can view class trends and see where students struggle with content and skills that will be assessed on the AP Exam. Students can view their own progress over time to improve their performance before the AP Exam.

AP QUESTION BANK

This online library of real AP Exam questions provides teachers with secure questions to use in their classrooms. Teachers can find questions indexed by course topics and skills, create customized tests, and assign them online or on paper. These tests enable students to practice and get feedback on each question.

* To report misuses, please call, 877-274-6474 (International: 212-632-1781).

Digital Activation

In order to teach an AP class and make sure students are registered to take the AP Exam, teachers must first complete the digital activation process. Digital activation gives students and teachers access to resources and gathers students' exam registration information online, eliminating most of the answer sheet bubbling that has added to testing time and fatigue.

AP teachers and students begin by signing in to **My AP** and completing a simple activation process at the start of the school year, which provides access to all AP resources, including AP Classroom.

To complete digital activation:

- Teachers and students sign in to or create their College Board accounts.
- Teachers confirm that they have added the course they teach to their AP Course Audit account and have had it approved by their school's administrator.
- Teachers or AP Coordinators, depending on who the school has decided is responsible, set up class sections so students can access AP resources and have exams ordered on their behalf.
- Students join class sections with a join code provided by their teacher or AP Coordinator.
- Students will be asked for additional registration information upon joining their first class section, which eliminates the need for extensive answer sheet bubbling on exam day.

While the digital activation process takes a short time for teachers, students, and AP Coordinators to complete, overall it helps save time and provides the following additional benefits:

- **Access to AP resources and supports:** Teachers have access to resources specifically designed to support instruction and provide feedback to students throughout the school year as soon as activation is complete.
- **Streamlined exam ordering:** AP Coordinators can create exam orders from the same online class rosters that enable students to access resources. The coordinator reviews, updates, and submits this information as the school's exam order in the fall.
- **Student registration labels:** For each student included in an exam order, schools will receive a set of personalized AP ID registration labels, which replaces the AP student pack. The AP ID connects a student's exam materials with the registration information they provided during digital activation, eliminating the need for pre-administration sessions and reducing time spent bubbling on exam day.
- **Targeted Instructional Planning Reports:** AP teachers will get Instructional Planning Reports (IPRs) that include data on each of their class sections automatically rather than relying on special codes optionally bubbled in on exam day.

Instructional Model

Integrating AP resources throughout the course can help students develop skills and conceptual understandings. The instructional model outlined below shows possible ways to incorporate AP resources into the classroom.

Plan

Teachers may consider the following approaches as they plan their instruction before teaching each unit.

- Review the overview at the start of each **unit guide** to identify essential questions, conceptual understandings, and skills for each unit.
- Use the **Unit at a Glance** table to identify related topics that build toward a common understanding, and then plan appropriate pacing for students.
- Identify useful strategies in the **Instructional Approaches** section to help teach the concepts and skills.

Teach

When teaching, supporting resources could be used to build students' conceptual understanding and their mastery of skills.

- Use the topic pages in the **unit guides** to identify the required content.
- Integrate the content with a skill, considering any appropriate scaffolding.
- Employ any of the instructional strategies previously identified.
- Use the available resources on the topic pages to bring a variety of assets into the classroom.

Assess

Teachers can measure student understanding of the content and skills covered in the unit and provide actionable feedback to students.

- At the end of each unit, use **AP Classroom** to assign students the online **Personal Progress Checks,** as homework or an in-class task.
- Provide question-level feedback to students through answer rationales; provide unit- and skill-level feedback using the progress dashboard.
- Create additional practice opportunities using the **AP Question Bank** and assign them through **AP Classroom**.

About the AP Calculus AB and BC Courses

AP Calculus AB and AP Calculus BC focus on students' understanding of calculus concepts and provide experience with methods and applications. Through the use of big ideas of calculus (e.g., modeling change, approximation and limits, and analysis of functions), each course becomes a cohesive whole, rather than a collection of unrelated topics. Both courses require students to use definitions and theorems to build arguments and justify conclusions.

The courses feature a multirepresentational approach to calculus, with concepts, results, and problems expressed graphically, numerically, analytically, and verbally. Exploring connections among these representations builds understanding of how calculus applies limits to develop important ideas, definitions, formulas, and theorems. A sustained emphasis on clear communication of methods, reasoning, justifications, and conclusions is essential. Teachers and students should regularly use technology to reinforce relationships among functions, to confirm written work, to implement experimentation, and to assist in interpreting results.

College Course Equivalent

AP Calculus AB is designed to be the equivalent of a first semester college calculus course devoted to topics in differential and integral calculus. AP Calculus BC is designed to be the equivalent to both first and second semester college calculus courses. AP Calculus BC applies the content and skills learned in AP Calculus AB to parametrically defined curves, polar curves, and vector-valued functions; develops additional integration techniques and applications; and introduces the topics of sequences and series.

Prerequisites

Before studying calculus, all students should complete the equivalent of four years of secondary mathematics designed for college-bound students: courses that should prepare them with a strong foundation in reasoning with algebraic symbols and working with algebraic structures. Prospective calculus students should take courses in which they study algebra, geometry, trigonometry, analytic geometry, and elementary functions. These functions include linear, polynomial, rational, exponential, logarithmic, trigonometric, inverse trigonometric, and piecewise-defined functions. In particular, before studying calculus, students must be familiar with the properties of functions, the composition of functions, the algebra of functions, and the graphs of functions.

Students must also understand the language of functions (domain and range, odd and even, periodic, symmetry, zeros, intercepts, and descriptors such as increasing and decreasing). Students should also know how the sine and cosine functions are defined from the unit circle and know the values of the trigonometric functions at the numbers $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$, and their multiples. Students who take AP Calculus BC should have basic familiarity with sequences and series, as well as some exposure to parametric and polar equations.

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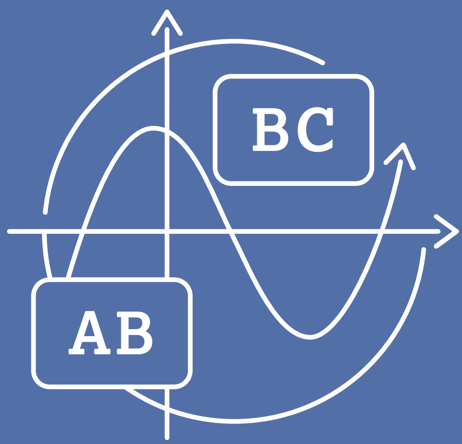
AP CALCULUS AB AND BC

Course Framework


BC
AB

Introduction

The course frameworks for AP Calculus AB and AP Calculus BC define content students must know and skills students must master in order to earn transferable, long-term understandings of calculus. The frameworks reflect a commitment to what college faculty value and mirror the curricula in corresponding college courses. Teachers may adjust the frameworks to meet state and local requirements.

The frameworks are organized into logical sequences, based on teacher input and commonly used textbooks. These sequences represent one reasonable learning pathway for each course, among many. Teachers may adjust the suggested sequencing of units or topics, although they will want to carefully consider how to account for such changes as they access course resources for planning, instruction, and assessment.

Balancing guidance and flexibility, this approach helps to prepare students for college credit and placement.

Course Framework Components

Overview

This course framework provides a clear and detailed description of the course requirements necessary for student success. The framework specifies what students should know, be able to do, and understand to qualify for college credit or placement.

The course framework includes two essential components:

1 MATHEMATICAL PRACTICES

The mathematical practices are central to the study and practice of calculus. Students should develop and apply the described skills on a regular basis over the span of the course.

2 COURSE CONTENT

The course content is organized into commonly taught units of study that provide a suggested sequence for the course. These units comprise the content and conceptual understandings that colleges and universities typically expect students to master to qualify for college credit and/or placement. This content is grounded in big ideas, which are cross-cutting concepts that build conceptual understanding and spiral throughout the course.

1

AP CALCULUS AB AND BC

Mathematical Practices

The AP Calculus AB and BC mathematical practices describe what a student should be able to do while exploring course concepts. The table that follows presents these practices, which students should develop during the AP Calculus AB and AP Calculus BC courses. These practices are categorized into skills, which form the basis of the tasks on the AP Exam.

The unit guides later in this publication embed and spiral these skills throughout the course, providing teachers with one way to integrate the skills in the course content with sufficient repetition to prepare students to transfer those skills when taking the AP Exam. Course content may be paired with a variety of skills on the AP Exam.

More detailed information about teaching the mathematical practices can be found in the Instructional Approaches section of this publication.

Mathematical Practices

Practice 1

Implementing Mathematical Processes 1

Determine expressions and values using mathematical procedures and rules.

Practice 2

Connecting Representations 2

Translate mathematical information from a single representation or across multiple representations.

Practice 3

Justification 3

Justify reasoning and solutions.

Practice 4

Communication and Notation 4

Use correct notation, language, and mathematical conventions to communicate results or solutions.

SKILLS

1.A Identify the question to be answered or problem to be solved *(not assessed)*.

1.B Identify key and relevant information to answer a question or solve a problem *(not assessed)*.

1.C Identify an appropriate mathematical rule or procedure based on the classification of a given expression (e.g., *Use the chain rule to find the derivative of a composite function*).

1.D Identify an appropriate mathematical rule or procedure based on the relationship between concepts (e.g., *rate of change and accumulation*) or processes (e.g., *differentiation and its inverse process, anti-differentiation*) to solve problems.

1.E Apply appropriate mathematical rules or procedures, with and without technology.

1.F Explain how an approximated value relates to the actual value.

2.A Identify common underlying structures in problems involving different contextual situations.

2.B Identify mathematical information from graphical, numerical, analytical, and/or verbal representations.

2.C Identify a re-expression of mathematical information presented in a given representation.

2.D Identify how mathematical characteristics or properties of functions are related in different representations.

2.E Describe the relationships among different representations of functions and their derivatives.

3.A Apply technology to develop claims and conjectures *(not assessed)*.

3.B Identify an appropriate mathematical definition, theorem, or test to apply.

3.C Confirm whether hypotheses or conditions of a selected definition, theorem, or test have been satisfied.

3.D Apply an appropriate mathematical definition, theorem, or test.

3.E Provide reasons or rationales for solutions and conclusions.

3.F Explain the meaning of mathematical solutions in context.

3.G Confirm that solutions are accurate and appropriate.

4.A Use precise mathematical language.

4.B Use appropriate units of measure.

4.C Use appropriate mathematical symbols and notation (e.g., *Represent a derivative using* $f'(x)$, y', *and* $\frac{dy}{dx}$).

4.D Use appropriate graphing techniques.

4.E Apply appropriate rounding procedures.

2

AP CALCULUS AB AND BC

Course Content

Based on the Understanding by Design® (Wiggins and McTighe) model, this course framework provides a clear and detailed description of the course requirements necessary for student success. The framework specifies what students must know, be able to do, and understand, with a focus on big ideas that encompass core principles, theories, and processes of the discipline. The framework also encourages instruction that prepares students for advanced coursework in mathematics or other fields engaged in modeling change (e.g., pure sciences, engineering, or economics) and for creating useful, reasonable solutions to problems encountered in an ever-changing world.

Big Ideas

The big ideas serve as the foundation of the course and allow students to create meaningful connections among concepts. They are often abstract concepts or themes that become threads that run throughout the course. Revisiting the big ideas and applying them in a variety of contexts allows students to develop deeper conceptual understanding. Below are the big ideas of the course and a brief description of each.

BIG IDEA 1: CHANGE (CHA)
Using derivatives to describe rates of change of one variable with respect to another or using definite integrals to describe the net change in one variable over an interval of another allows students to understand change in a variety of contexts. It is critical that students grasp the relationship between integration and differentiation as expressed in the Fundamental Theorem of Calculus—a central idea in AP Calculus.

BIG IDEA 2: LIMITS (LIM)
Beginning with a discrete model and then considering the consequences of a limiting case allows us to model real-world behavior and to discover and understand important ideas, definitions, formulas, and theorems in calculus: for example, continuity, differentiation, integration, and series **BC ONLY**.

BIG IDEA 3: ANALYSIS OF FUNCTIONS (FUN)
Calculus allows us to analyze the behaviors of functions by relating limits to differentiation, integration, and infinite series and relating each of these concepts to the others.

UNITS

The course content is organized into commonly taught units. The units have been arranged in a logical sequence frequently found in many college courses and textbooks.

The eight units in AP Calculus AB and ten units in AP Calculus BC, and their weighting on the multiple-choice section of the AP Exam, are listed on the following page.

Pacing recommendations at the unit level and on the Course at a Glance provide suggestions for how teachers can teach the required course content and administer the Personal Progress Checks.

The suggested class periods are based on a schedule in which the class meets five days a week for 45 minutes each day. While these recommendations have been made to aid planning, teachers are of course free to adjust the pacing based on the needs of their students, alternate schedules (e.g., block scheduling), or their school's academic calendar.

TOPICS

Each unit is broken down into teachable segments called topics. The topic pages (starting on p. 35) contain the required content for each topic. Although most topics can be taught in one or two class periods, teachers should pace the course to suit the needs of their students and school.

Exam Weighting for the Multiple-Choice Section of the AP Exam

Units	Exam Weighting (AB)	Exam Weighting (BC)
Unit 1: Limits and Continuity	**10–12%**	**4–7%**
Unit 2: Differentiation: Definition and Fundamental Properties	**10–12%**	**4–7%**
Unit 3: Differentiation: Composite, Implicit, and Inverse Functions	**9–13%**	**4–7%**
Unit 4: Contextual Applications of Differentiation	**10–15%**	**6–9%**
Unit 5: Analytical Applications of Differentiation	**15–18%**	**8–11%**
Unit 6: Integration and Accumulation of Change	**17–20%**	**17–20%**
Unit 7: Differential Equations	**6–12%**	**6–9%**
Unit 8: Applications of Integration	**10–15%**	**6–9%**
Unit 9: Parametric Equations, Polar Coordinates, and Vector-Valued Functions **BC ONLY**		**11–12%**
Unit 10: Infinite Sequences and Series **BC ONLY**		**17–18%**

Spiraling the Big Ideas

The following table shows how the big ideas spiral across units.

Big Ideas	Unit 1 Limits and Continuity	Unit 2 Differentiation: Definition and Fundamental Properties	Unit 3 Differentiation: Composite, Implicit, and Inverse Functions	Unit 4 Contextual Applications of Differentiation	Unit 5 Analytical Applications of Differentiation
Change CHA	✓	✓		✓	
Limits LIM	✓	✓		✓	
Analysis of Functions FUN	✓	✓	✓		✓

Spiraling the Big Ideas *(cont'd)*

Big Ideas	Unit 6	Unit 7	Unit 8	Unit 9	Unit 10
	Integration and Accumulation of Change	Differential Equations	Applications of Integration	Parametric Equations, Polar Coordinates, and Vector-Valued Functions **BC ONLY**	Infinite Sequences and Series **BC ONLY**
Change CHA	✓		✓	✓	
Limits LIM	✓				✓
Analysis of Functions FUN	✓	✓		✓	

Course at a Glance

Plan

The Course at a Glance provides a useful visual organization of the AP Calculus AB and AP Calculus BC curricular components, including:

- Sequence of units, along with approximate weighting and suggested pacing. Please note, pacing is based on 45-minute class periods, meeting five days each week for a full academic year.
- Progression of topics within each unit.
- Spiraling of the big ideas and mathematical practices across units.

Teach

MATHEMATICAL PRACTICES

Mathematical practices spiral throughout the course.

- 1 Implementing Mathematical Processes
- 2 Connecting Representations
- 3 Justification
- 4 Communication and Notation

BIG IDEAS

Big ideas spiral across topics and units.

- CHA Change
- LIM Limits
- FUN Analysis of Functions

BC ONLY

The purple shading represents BC only content.

Assess

Assign the Personal Progress Checks—either as homework or in class—for each unit. Each Personal Progress Check contains formative multiple-choice and free-response questions. The feedback from the Personal Progress Checks shows students the areas where they need to focus.

UNIT 1 Limits and Continuity

AP EXAM WEIGHTING: 10–12% AB, 4–7% BC

CLASS PERIODS: ~22–23 AB, ~13–14 BC

Big Idea	Practice	Topic
CHA	2	1.1 Introducing Calculus: Can Change Occur at an Instant?
LIM	2	1.2 Defining Limits and Using Limit Notation
LIM	2	1.3 Estimating Limit Values from Graphs
LIM	2	1.4 Estimating Limit Values from Tables
LIM	1	1.5 Determining Limits Using Algebraic Properties of Limits
LIM	1	1.6 Determining Limits Using Algebraic Manipulation
LIM	1	1.7 Selecting Procedures for Determining Limits
LIM	3	1.8 Determining Limits Using the Squeeze Theorem
LIM	2	1.9 Connecting Multiple Representations of Limits
LIM	3	1.10 Exploring Types of Discontinuities
LIM	3	1.11 Defining Continuity at a Point
LIM	1	1.12 Confirming Continuity over an Interval
LIM	1	1.13 Removing Discontinuities
LIM	3	1.14 Connecting Infinite Limits and Vertical Asymptotes
LIM	2	1.15 Connecting Limits at Infinity and Horizontal Asymptotes
FUN	3	1.16 Working with the Intermediate Value Theorem (IVT)

Personal Progress Check 1

Multiple-choice: ~45 questions

Free-response: 3 questions (partial)

UNIT 2 Differentiation: Definition and Basic Derivative Rules

AP EXAM WEIGHTING: 10–12% AB, 4–7% BC

CLASS PERIODS: ~13–14 AB, ~9–10 BC

Big Idea	Practice	Topic
CHA	2	2.1 Defining Average and Instantaneous Rates of Change at a Point
CHA	1, 4	2.2 Defining the Derivative of a Function and Using Derivative Notation
CHA	1	2.3 Estimating Derivatives of a Function at a Point
FUN	3	2.4 Connecting Differentiability and Continuity: Determining When Derivatives Do and Do Not Exist
FUN	1	2.5 Applying the Power Rule
FUN	1	2.6 Derivative Rules: Constant, Sum, Difference, and Constant Multiple
FUN, LIM	1	2.7 Derivatives of cos x, sin x, e^x, and ln x
FUN	1	2.8 The Product Rule
FUN	1	2.9 The Quotient Rule
FUN	1	2.10 Finding the Derivatives of Tangent, Cotangent, Secant, and/or Cosecant Functions

Personal Progress Check 2

Multiple-choice: ~30 questions

Free-response: 3 questions (partial)

NOTE: Partial versions of the free-response questions are provided to prepare students for more complex, full questions that they will encounter on the AP Exam.

UNIT 3 Differentiation: Composite, Implicit, and Inverse Functions

AP EXAM WEIGHTING 9–13% AB 4–7% BC

CLASS PERIODS ~10–11 AB ~8–9 BC

- FUN 1 — 3.1 The Chain Rule
- FUN 1 — 3.2 Implicit Differentiation
- FUN 3 — 3.3 Differentiating Inverse Functions
- FUN 1 — 3.4 Differentiating Inverse Trigonometric Functions
- FUN 1 — 3.5 Selecting Procedures for Calculating Derivatives
- FUN 1 — 3.6 Calculating Higher-Order Derivatives

Personal Progress Check 3

Multiple-choice: ~15 questions
Free-response: 3 questions (partial/full)

UNIT 4 Contextual Applications of Differentiation

AP EXAM WEIGHTING 10–15% AB 6–9% BC

CLASS PERIODS ~10–11 AB ~6–7 BC

- CHA 1 — 4.1 Interpreting the Meaning of the Derivative in Context
- CHA 1 — 4.2 Straight-Line Motion: Connecting Position, Velocity, and Acceleration
- CHA 2 — 4.3 Rates of Change in Applied Contexts Other Than Motion
- CHA 1 — 4.4 Introduction to Related Rates
- CHA 3 — 4.5 Solving Related Rates Problems
- CHA 1 — 4.6 Approximating Values of a Function Using Local Linearity and Linearization
- LIM 3 — 4.7 Using L'Hospital's Rule for Determining Limits of Indeterminate Forms

Personal Progress Check 4

Multiple-choice: ~15 questions
Free-response: 3 questions

UNIT 5 Analytical Applications of Differentiation

AP EXAM WEIGHTING 15–18% AB 8–11% BC

CLASS PERIODS ~15–16 AB ~10–11 BC

- FUN 3 — 5.1 Using the Mean Value Theorem
- FUN 3 — 5.2 Extreme Value Theorem, Global Versus Local Extrema, and Critical Points
- FUN 2 — 5.3 Determining Intervals on Which a Function Is Increasing or Decreasing
- FUN 3 — 5.4 Using the First Derivative Test to Determine Relative (Local) Extrema
- FUN 1 — 5.5 Using the Candidates Test to Determine Absolute (Global) Extrema
- FUN 2 — 5.6 Determining Concavity of Functions over Their Domains
- FUN 3 — 5.7 Using the Second Derivative Test to Determine Extrema
- FUN 2 — 5.8 Sketching Graphs of Functions and Their Derivatives
- FUN 2 — 5.9 Connecting a Function, Its First Derivative, and Its Second Derivative
- FUN 2 — 5.10 Introduction to Optimization Problems
- FUN 3 — 5.11 Solving Optimization Problems
- FUN 1 3 — 5.12 Exploring Behaviors of Implicit Relations

Personal Progress Check 5

Multiple-choice: ~35 questions
Free-response: 3 questions

UNIT 6 Integration and Accumulation of Change

AP EXAM WEIGHTING	17–20% AB	17–20% BC
CLASS PERIODS	~18–20 AB	~15–16 BC

Skill	Topic	
CHA 4	6.1	Exploring Accumulations of Change
LIM 1	6.2	Approximating Areas with Riemann Sums
LIM 2	6.3	Riemann Sums, Summation Notation, and Definite Integral Notation
FUN 1	6.4	The Fundamental Theorem of Calculus and Accumulation Functions
FUN 2	6.5	Interpreting the Behavior of Accumulation Functions Involving Area
FUN 3	6.6	Applying Properties of Definite Integrals
FUN 3	6.7	The Fundamental Theorem of Calculus and Definite Integrals
FUN 4	6.8	Finding Antiderivatives and Indefinite Integrals: Basic Rules and Notation
FUN 1	6.9	Integrating Using Substitution
FUN 1	6.10	Integrating Functions Using Long Division and Completing the Square
FUN 1	6.11	Integrating Using Integration by Parts BC ONLY
FUN 1	6.12	Using Linear Partial Fractions BC ONLY
LIM 1	6.13	Evaluating Improper Integrals BC ONLY
FUN 1	6.14	Selecting Techniques for Antidifferentiation

Personal Progress Check 6

Multiple-choice:
- ~25 questions (AB)
- ~35 questions (BC)

Free-response: 3 questions

UNIT 7 Differential Equations

AP EXAM WEIGHTING	6–12% AB	6–9% BC
CLASS PERIODS	~8–9 AB	~9–10 BC

Skill	Topic	
FUN 2	7.1	Modeling Situations with Differential Equations
FUN 3	7.2	Verifying Solutions for Differential Equations
FUN 2	7.3	Sketching Slope Fields
FUN 4	7.4	Reasoning Using Slope Fields
FUN 1	7.5	Approximating Solutions Using Euler's Method BC ONLY
FUN 1	7.6	Finding General Solutions Using Separation of Variables
FUN 1	7.7	Finding Particular Solutions Using Initial Conditions and Separation of Variables
FUN 3	7.8	Exponential Models with Differential Equations
FUN 3	7.9	Logistic Models with Differential Equations BC ONLY

Personal Progress Check 7

Multiple-choice:
- ~15 questions (AB)
- ~20 questions (BC)

Free-response: 3 questions

UNIT 8 Applications of Integration

AP EXAM WEIGHTING	10–15% AB	6–9% BC
CLASS PERIODS	~19–20 AB	~13–14 BC

Skill	Topic	
CHA 1	8.1	Finding the Average Value of a Function on an Interval
CHA 1	8.2	Connecting Position, Velocity, and Acceleration of Functions Using Integrals
CHA 3	8.3	Using Accumulation Functions and Definite Integrals in Applied Contexts
CHA 4	8.4	Finding the Area Between Curves Expressed as Functions of x
CHA 1	8.5	Finding the Area Between Curves Expressed as Functions of y
CHA 2	8.6	Finding the Area Between Curves That Intersect at More Than Two Points
CHA 3	8.7	Volumes with Cross Sections: Squares and Rectangles
CHA 3	8.8	Volumes with Cross Sections: Triangles and Semicircles
CHA 3	8.9	Volume with Disc Method: Revolving Around the x- or y-Axis
CHA 2	8.10	Volume with Disc Method: Revolving Around Other Axes
CHA 4	8.11	Volume with Washer Method: Revolving Around the x- or y-Axis
CHA 2	8.12	Volume with Washer Method: Revolving Around Other Axes
CHA 3	8.13	The Arc Length of a Smooth, Planar Curve and Distance Traveled BC ONLY

Personal Progress Check 8

Multiple-choice: ~30 questions
Free-response: 3 questions

UNIT 9 Parametric Equations, Polar Coordinates, and Vector-Valued Functions BC ONLY

AP EXAM WEIGHTING N/A AB 11–12% BC

CLASS PERIODS N/A AB ~10–11 BC

CHA 2	9.1	Defining and Differentiating Parametric Equations
CHA 1	9.2	Second Derivatives of Parametric Equations
CHA 1	9.3	Finding Arc Lengths of Curves Given by Parametric Equations
CHA 1	9.4	Defining and Differentiating Vector-Valued Functions
FUN 1	9.5	Integrating Vector-Valued Functions
FUN 1	9.6	Solving Motion Problems Using Parametric and Vector-Valued Functions
FUN 2	9.7	Defining Polar Coordinates and Differentiating in Polar Form
CHA 3	9.8	Find the Area of a Polar Region or the Area Bounded by a Single Polar Curve
CHA 3	9.9	Finding the Area of the Region Bounded by Two Polar Curves

Personal Progress Check 9

Multiple-choice: ~25 questions
Free-response: 3 questions

UNIT 10 Infinite Sequences and Series BC ONLY

AP EXAM WEIGHTING N/A AB 17–18% BC

CLASS PERIODS N/A AB ~17–18 BC

LIM 3	10.1	Defining Convergent and Divergent Infinite Series
LIM 3	10.2	Working with Geometric Series
LIM 3	10.3	The nth Term Test for Divergence
LIM 3	10.4	Integral Test for Convergence
LIM 3	10.5	Harmonic Series and p-Series
LIM 3	10.6	Comparison Tests for Convergence
LIM 3	10.7	Alternating Series Test for Convergence
LIM 3	10.8	Ratio Test for Convergence
LIM 3	10.9	Determining Absolute or Conditional Convergence
LIM 1	10.10	Alternating Series Error Bound
LIM 3 2	10.11	Finding Taylor Polynomial Approximations of Functions
LIM 1	10.12	Lagrange Error Bound
LIM 2	10.13	Radius and Interval of Convergence of Power Series
LIM 2	10.14	Finding Taylor or Maclaurin Series for a Function
LIM 3	10.15	Representing Functions as Power Series

Personal Progress Check 10

Multiple-choice: ~45 questions
Free-response: 3 questions

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AP CALCULUS AB AND BC

Unit Guides

Introduction

Designed with extensive input from the community of AP Calculus AB and AP Calculus BC educators, the unit guides offer teachers helpful guidance in building students' skills and knowledge. The suggested sequence was identified through a thorough analysis of the syllabi of highly effective AP teachers and the organization of typical college textbooks.

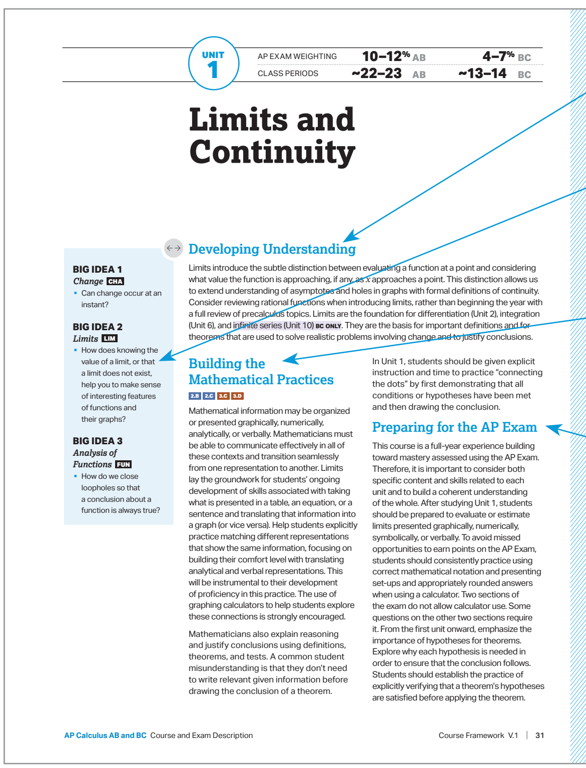
This unit structure respects new AP teachers' time by providing one possible sequence they can adopt or modify rather than having to build from scratch. An additional benefit is that these units enable the AP Program to provide interested teachers with formative assessments—the Personal Progress Checks—that they can assign their students at the end of each unit to gauge progress toward success on the AP Exam. However, experienced AP teachers who are satisfied with their current course organization and exam results should feel no pressure to adopt these units, which comprise an optional sequence for this course.

REQUIRED COURSE CONTENT LABELING SYSTEM

BIG IDEA	ENDURING UNDERSTANDING	LEARNING OBJECTIVE	ESSENTIAL KNOWLEDGE
LIM →	**LIM-1** →	**LIM-1.B** →	**LIM-1.B.1**
Limits	Reasoning with definitions, theorems, and properties can be used to justify claims about limits.	Interpret limits expressed in analytic notation.	A limit can be expressed in multiple ways, including graphically, numerically, and analytically.

NOTE: Labels are used to distinguish each unique element of the required course content and are used throughout this course and exam description. Additionally, they are used in the AP Question Bank and other resources found in AP Classroom. Enduring understandings are labeled sequentially according to the big idea that they are related to. Learning objectives are labeled to correspond with the enduring understanding they relate to. Finally, essential knowledge statements are labeled to correspond with the learning objective they relate to.

Using the Unit Guides


UNIT 1

AP EXAM WEIGHTING	10–12%	AB	4–7%	BC
CLASS PERIODS	~22–23	AB	~13–14	BC

Limits and Continuity

BIG IDEA 1
Change CHA
- Can change occur at an instant?

BIG IDEA 2
Limits LIM
- How does knowing the value of a limit, or that a limit does not exist, help you to make sense of interesting features of functions and their graphs?

BIG IDEA 3
Analysis of Functions FUN
- How do we close loopholes so that a conclusion about a function is always true?

Developing Understanding

Limits introduce the subtle distinction between evaluating a function at a point and considering what value the function is approaching, if any, as x approaches a point. This distinction allows us to extend understanding of asymptotes and holes in graphs with formal definitions of continuity. Consider reviewing rational functions when introducing limits, rather than beginning the year with a full review of precalculus topics. Limits are the foundation for differentiation (Unit 2), integration (Unit 6), and infinite series (Unit 10) BC ONLY. They are the basis for important definitions and for theorems that are used to solve realistic problems involving change and to justify conclusions.

Building the Mathematical Practices

2.B 2.C 3.C 3.D

Mathematical information may be organized or presented graphically, numerically, analytically, or verbally. Mathematicians must be able to communicate effectively in all of these contexts and transition seamlessly from one representation to another. Limits lay the groundwork for students' ongoing development of skills associated with taking what is presented in a table, an equation, or a sentence and translating that information into a graph (or vice versa). Help students explicitly practice matching different representations that show the same information, focusing on building their comfort level with translating analytical and verbal representations. This will be instrumental to their development of proficiency in this practice. The use of graphing calculators to help students explore these connections is strongly encouraged.

Mathematicians also explain reasoning and justify conclusions using definitions, theorems, and tests. A common student misunderstanding is that they don't need to write relevant given information before drawing the conclusion of a theorem.

In Unit 1, students should be given explicit instruction and time to practice "connecting the dots" by first demonstrating that all conditions or hypotheses have been met and then drawing the conclusion.

Preparing for the AP Exam

This course is a full-year experience building toward mastery assessed using the AP Exam. Therefore, it is important to consider both specific content and skills related to each unit and to build a coherent understanding of the whole. After studying Unit 1, students should be prepared to evaluate or estimate limits presented graphically, numerically, symbolically, or verbally. To avoid missed opportunities to earn points on the AP Exam, students should consistently practice using correct mathematical notation and presenting set-ups and appropriately rounded answers when using a calculator. Two sections of the exam do not allow calculator use. Some questions on the other two sections require it. From the first unit onward, emphasize the importance of hypotheses for theorems. Explore why each hypothesis is needed in order to ensure that the conclusion follows. Students should establish the practice of explicitly verifying that a theorem's hypotheses are satisfied before applying the theorem.

AP Calculus AB and BC Course and Exam Description

Course Framework V.1 | 31

UNIT OPENERS

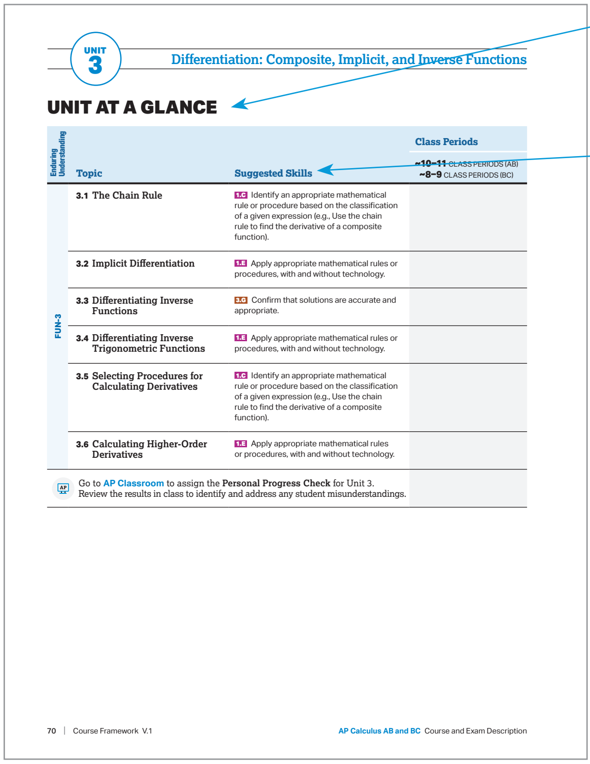
Developing Understanding provides an overview that contextualizes and situates the key content of the unit within the scope of the course.

The **big ideas** serve as the foundation of the course and help develop understanding as they spiral throughout the course. The **essential questions** are thought-provoking questions that motivate students and inspire inquiry.

Building the Mathematical Practices describes specific skills within the practices that are appropriate to focus on in that unit. Certain practices have been noted to indicate areas of emphasis for that unit.

Preparing for the AP Exam provides helpful tips and common student misunderstandings identified from prior exam data.

Using the Unit Guides



UNIT 3 — Differentiation: Composite, Implicit, and Inverse Functions

UNIT AT A GLANCE

Enduring Understanding	Topic	Suggested Skills	Class Periods ~10–11 CLASS PERIODS (AB) ~8–9 CLASS PERIODS (BC)
FUN-3	**3.1 The Chain Rule**	1.C Identify an appropriate mathematical rule or procedure based on the classification of a given expression (e.g., Use the chain rule to find the derivative of a composite function).	
	3.2 Implicit Differentiation	1.E Apply appropriate mathematical rules or procedures, with and without technology.	
	3.3 Differentiating Inverse Functions	3.G Confirm that solutions are accurate and appropriate.	
	3.4 Differentiating Inverse Trigonometric Functions	1.E Apply appropriate mathematical rules or procedures, with and without technology.	
	3.5 Selecting Procedures for Calculating Derivatives	1.C Identify an appropriate mathematical rule or procedure based on the classification of a given expression (e.g., Use the chain rule to find the derivative of a composite function).	
	3.6 Calculating Higher-Order Derivatives	1.E Apply appropriate mathematical rules or procedures, with and without technology.	

Go to **AP Classroom** to assign the **Personal Progress Check** for Unit 3. Review the results in class to identify and address any student misunderstandings.

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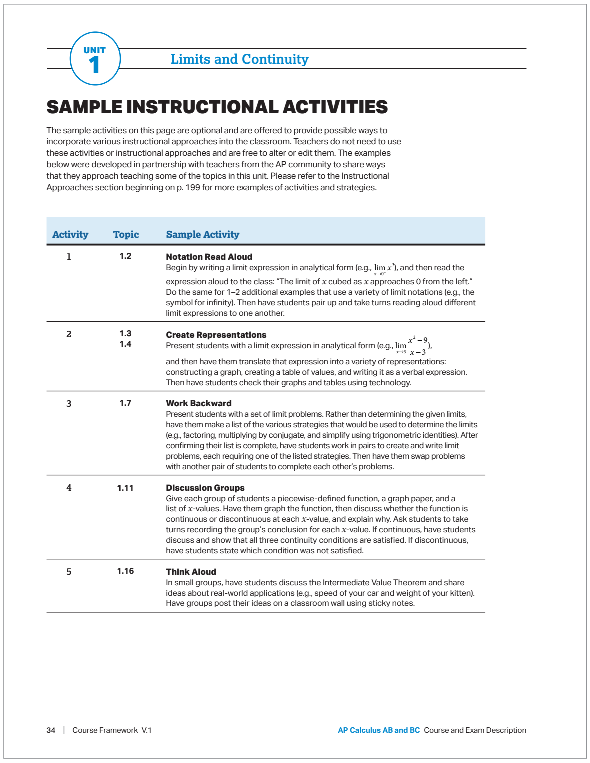
The **Unit at a Glance** table shows the topics, related enduring understandings, and suggested skills. The class periods column has been left blank so that teachers can customize the time they spend on each topic.

The **suggested skills** for each topic show possible ways to link the content in that topic to specific AP Calculus AB and/or BC skills. The individual skills have been thoughtfully chosen in a way that scaffolds the skills throughout the course. The questions on the Personal Progress Checks are based on this pairing. However, AP Exam questions can pair the content with any of the skills.



UNIT 1 — Limits and Continuity

SAMPLE INSTRUCTIONAL ACTIVITIES

The sample activities on this page are optional and are offered to provide possible ways to incorporate various instructional approaches into the classroom. Teachers do not need to use these activities or instructional approaches and are free to alter or edit them. The examples below were developed in partnership with teachers from the AP community to share ways that they approach teaching some of the topics in this unit. Please refer to the Instructional Approaches section beginning on p. 199 for more examples of activities and strategies.

Activity	Topic	Sample Activity
1	1.2	**Notation Read Aloud** Begin by writing a limit expression in analytical form (e.g., $\lim_{x\to 0^-} x^3$), and then read the expression aloud to the class: "The limit of x cubed as x approaches 0 from the left." Do the same for 1–2 additional examples that use a variety of limit notations (e.g., the symbol for infinity). Then have students pair up and take turns reading aloud different limit expressions to one another.
2	1.3 1.4	**Create Representations** Present students with a limit expression in analytical form (e.g., $\lim_{x\to 3} \frac{x^2-9}{x-3}$), and then have them translate that expression into a variety of representations: constructing a graph, creating a table of values, and writing it as a verbal expression. Then have students check their graphs and tables using technology.
3	1.7	**Work Backward** Present students with a set of limit problems. Rather than determining the given limits, have them make a list of the various strategies that would be used to determine the limits (e.g., factoring, multiplying by conjugate, and simplify using trigonometric identities). After confirming their list is complete, have students work in pairs to create and write limit problems, each requiring one of the listed strategies. Then have them swap problems with another pair of students to complete each other's problems.
4	1.11	**Discussion Groups** Give each group of students a piecewise-defined function, a graph paper, and a list of x-values. Have them graph the function, then discuss whether the function is continuous or discontinuous at each x-value, and explain why. Ask students to take turns recording the group's conclusion for each x-value. If continuous, have students discuss and show that all three continuity conditions are satisfied. If discontinuous, have students state which condition was not satisfied.
5	1.16	**Think Aloud** In small groups, have students discuss the Intermediate Value Theorem and share ideas about real-world applications (e.g., speed of your car and weight of your kitten). Have groups post their ideas on a classroom wall using sticky notes.

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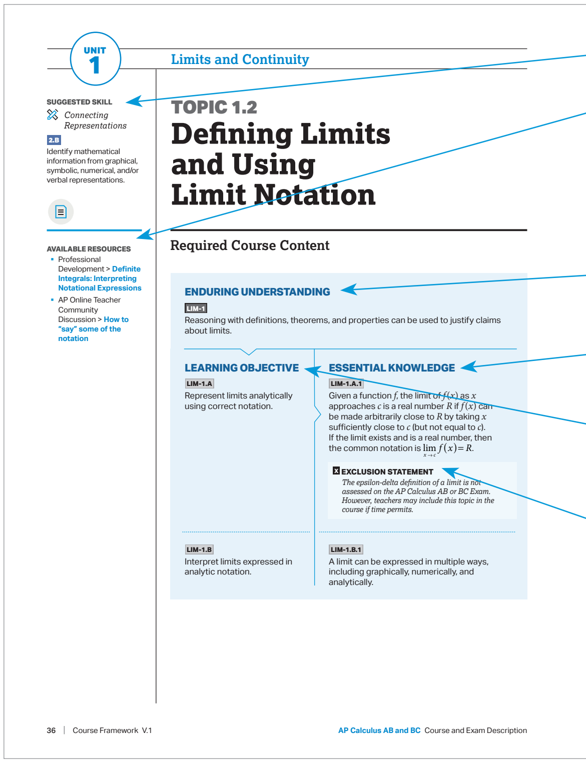
The **Sample Instructional Activities** page includes optional activities that can help teachers tie together the content and skill for a particular topic.

Using the Unit Guides


UNIT 1

Limits and Continuity

SUGGESTED SKILL

Connecting Representations

2.B Identify mathematical information from graphical, symbolic, numerical, and/or verbal representations.

AVAILABLE RESOURCES
- Professional Development > Definite Integrals: Interpreting Notational Expressions
- AP Online Teacher Community Discussion > How to "say" some of the notation

TOPIC 1.2

Defining Limits and Using Limit Notation

Required Course Content

ENDURING UNDERSTANDING

LIM-1

Reasoning with definitions, theorems, and properties can be used to justify claims about limits.

LEARNING OBJECTIVE	ESSENTIAL KNOWLEDGE
LIM-1.A Represent limits analytically using correct notation.	LIM-1.A.1 Given a function f, the limit of $f(x)$ as x approaches c is a real number R if $f(x)$ can be made arbitrarily close to R by taking x sufficiently close to c (but not equal to c). If the limit exists and is a real number, then the common notation is $\lim_{x \to c} f(x) = R$. EXCLUSION STATEMENT *The epsilon-delta definition of a limit is not assessed on the AP Calculus AB or BC Exam. However, teachers may include this topic in the course if time permits.*
LIM-1.B Interpret limits expressed in analytic notation.	LIM-1.B.1 A limit can be expressed in multiple ways, including graphically, numerically, and analytically.

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TOPIC PAGES

The **suggested skill** offers a possible skill to pair with the topic.

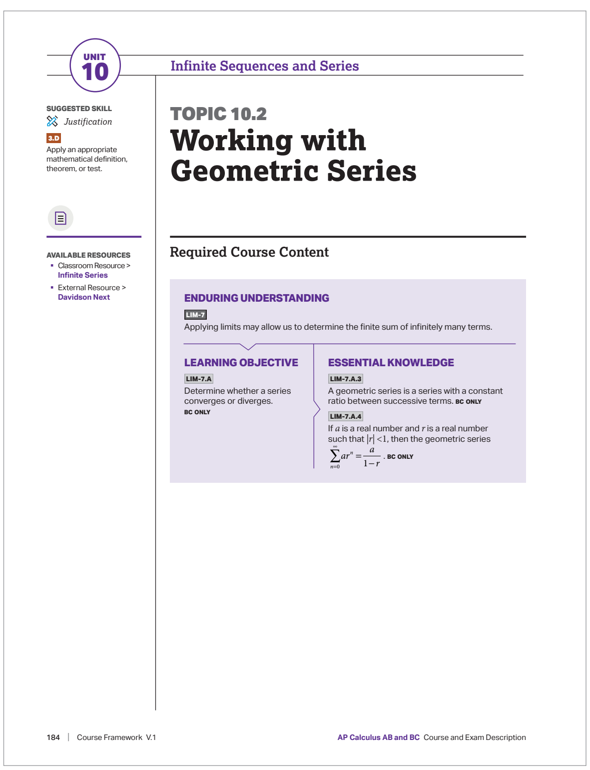
Where possible, **available resources** are provided that might help address a particular topic. Some of these resources may have titles that refer to different topics, but those resources are listed for various units because they also model strategies that are applicable across multiple topics.

Enduring understandings are the long-term takeaways related to the big ideas that leave a lasting impression on students.

Essential knowledge statements describe the knowledge required to perform the learning objective.

Learning objectives define what a student needs to be able to do with content knowledge in order to progress toward the enduring understandings.

Exclusion statements identify topics that may be covered in a first-year college calculus course but are not assessed on the AP Calculus AB or BC Exam. Although these topics are not assessed, the AP Calculus courses are designed to support teachers who wish to introduce these topics to students.


UNIT 10

Infinite Sequences and Series

SUGGESTED SKILL

Justification

3.D Apply an appropriate mathematical definition, theorem, or test.

AVAILABLE RESOURCES
- Classroom Resource > Infinite Series
- External Resource > Davidson Next

TOPIC 10.2

Working with Geometric Series

Required Course Content

ENDURING UNDERSTANDING

LIM-7

Applying limits may allow us to determine the finite sum of infinitely many terms.

LEARNING OBJECTIVE	ESSENTIAL KNOWLEDGE
LIM-7.A Determine whether a series converges or diverges. BC ONLY	LIM-7.A.3 A geometric series is a series with a constant ratio between successive terms. BC ONLY LIM-7.A.4 If a is a real number and r is a real number such that $\lvert r \rvert < 1$, then the geometric series $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$. BC ONLY

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BC ONLY

Page elements that have purple shading represent BC only content. Topics, learning objectives, essential knowledge statements, and content in the unit openers that apply to AP Calculus BC only are shaded and labeled **BC ONLY**. Note that all of Units 9 and 10 apply to AP Calculus BC only.

AP CALCULUS AB AND BC

UNIT 1

Limits and Continuity

AP EXAM WEIGHTING	10–12% AB	4–7% BC
CLASS PERIODS	~22–23 AB	~13–14 BC

Remember to go to **AP Classroom** to assign students the online **Personal Progress Check** for this unit.

Whether assigned as homework or completed in class, the **Personal Progress Check** provides each student with immediate feedback related to this unit's topics and skills.

Personal Progress Check 1

Multiple-choice: ~45 questions
Free-response: 3 questions (partial)

UNIT 1

AP EXAM WEIGHTING	10–12%	AB	4–7%	BC
CLASS PERIODS	~22–23	AB	~13–14	BC

Limits and Continuity

BIG IDEA 1
Change **CHA**
- Can change occur at an instant?

BIG IDEA 2
Limits **LIM**
- How does knowing the value of a limit, or that a limit does not exist, help you to make sense of interesting features of functions and their graphs?

BIG IDEA 3
Analysis of Functions **FUN**
- How do we close loopholes so that a conclusion about a function is always true?

Developing Understanding

Limits introduce the subtle distinction between evaluating a function at a point and considering what value the function is approaching, if any, as x approaches a point. This distinction allows us to extend understanding of asymptotes and holes in graphs with formal definitions of continuity. Consider reviewing rational functions when introducing limits, rather than beginning the year with a full review of precalculus topics. Limits are the foundation for differentiation (Unit 2), integration (Unit 6), and infinite series (Unit 10) **BC ONLY**. They are the basis for important definitions and for theorems that are used to solve realistic problems involving change and to justify conclusions.

Building the Mathematical Practices

2.B 2.C 3.C 3.D

Mathematical information may be organized or presented graphically, numerically, analytically, or verbally. Mathematicians must be able to communicate effectively in all of these contexts and transition seamlessly from one representation to another. Limits lay the groundwork for students' ongoing development of skills associated with taking what is presented in a table, an equation, or a sentence and translating that information into a graph (or vice versa). Help students explicitly practice matching different representations that show the same information, focusing on building their comfort level with translating analytical and verbal representations. This will be instrumental to their development of proficiency in this practice. The use of graphing calculators to help students explore these connections is strongly encouraged.

Mathematicians also explain reasoning and justify conclusions using definitions, theorems, and tests. A common student misunderstanding is that they don't need to write relevant given information before drawing the conclusion of a theorem.

In Unit 1, students should be given explicit instruction and time to practice "connecting the dots" by first demonstrating that all conditions or hypotheses have been met and then drawing the conclusion.

Preparing for the AP Exam

This course is a full-year experience building toward mastery assessed using the AP Exam. Therefore, it is important to consider both specific content and skills related to each unit and to build a coherent understanding of the whole. After studying Unit 1, students should be prepared to evaluate or estimate limits presented graphically, numerically, analytically, or verbally. To avoid missed opportunities to earn points on the AP Exam, students should consistently practice using correct mathematical notation and presenting set-ups and appropriately rounded answers when using a calculator. Two sections of the exam do not allow calculator use. Some questions on the other two sections require it. From the first unit onward, emphasize the importance of hypotheses for theorems. Explore why each hypothesis is needed in order to ensure that the conclusion follows. Students should establish the practice of explicitly verifying that a theorem's hypotheses are satisfied before applying the theorem.

UNIT AT A GLANCE

Enduring Understanding	Topic	Suggested Skills	Class Periods ~22–23 CLASS PERIODS (AB) ~13–14 CLASS PERIODS (BC)
CHA-1	1.1 Introducing Calculus: Can Change Occur at an Instant?	2.B Identify mathematical information from graphical, numerical, analytical, and/or verbal representations.	
LIM-1	1.2 Defining Limits and Using Limit Notation	2.B Identify mathematical information from graphical, numerical, analytical, and/or verbal representations.	
	1.3 Estimating Limit Values from Graphs	2.B Identify mathematical information from graphical, numerical, analytical, and/or verbal representations.	
	1.4 Estimating Limit Values from Tables	2.B Identify mathematical information from graphical, numerical, analytical, and/or verbal representations.	
	1.5 Determining Limits Using Algebraic Properties of Limits	1.E Apply appropriate mathematical rules or procedures, with and without technology.	
	1.6 Determining Limits Using Algebraic Manipulation	1.C Identify an appropriate mathematical rule or procedure based on the classification of a given expression (e.g., Use the chain rule to find the derivative of a composite function).	
	1.7 Selecting Procedures for Determining Limits	1.C Identify an appropriate mathematical rule or procedure based on the classification of a given expression (e.g., Use the chain rule to find the derivative of a composite function).	
	1.8 Determining Limits Using the Squeeze Theorem	3.C Confirm whether hypotheses or conditions of a selected definition, theorem, or test have been satisfied.	
	1.9 Connecting Multiple Representations of Limits	2.C Identify a re-expression of mathematical information presented in a given representation.	

continued on next page

UNIT AT A GLANCE *(cont'd)*

Enduring Understanding	Topic	Suggested Skills	Class Periods ~22–23 CLASS PERIODS (AB) ~13–14 CLASS PERIODS (BC)
LIM-2	**1.10 Exploring Types of Discontinuities**	3.B Identify an appropriate mathematical definition, theorem, or test to apply.	
	1.11 Defining Continuity at a Point	3.C Confirm whether hypotheses or conditions of a selected definition, theorem, or test have been satisfied.	
	1.12 Confirming Continuity over an Interval	1.E Apply appropriate mathematical rules or procedures, with and without technology.	
	1.13 Removing Discontinuities	1.E Apply appropriate mathematical rules or procedures, with and without technology.	
	1.14 Connecting Infinite Limits and Vertical Asymptotes	3.D Apply an appropriate mathematical definition, theorem, or test.	
	1.15 Connecting Limits at Infinity and Horizontal Asymptotes	2.D Identify how mathematical characteristics or properties of functions are related in different representations.	
FUN-1	**1.16 Working with the Intermediate Value Theorem (IVT)**	3.E Provide reasons or rationales for solutions or conclusions.	

Go to **AP Classroom** to assign the **Personal Progress Check** for Unit 1. Review the results in class to identify and address any student misunderstandings.

SAMPLE INSTRUCTIONAL ACTIVITIES

The sample activities on this page are optional and are offered to provide possible ways to incorporate various instructional approaches into the classroom. Teachers do not need to use these activities or instructional approaches and are free to alter or edit them. The examples below were developed in partnership with teachers from the AP community to share ways that they approach teaching some of the topics in this unit. Please refer to the Instructional Approaches section beginning on p. 199 for more examples of activities and strategies.

Activity	Topic	Sample Activity
1	1.2	**Notation Read Aloud** Begin by writing a limit expression in analytical form (e.g., $\lim\limits_{x\to 0^-} x^3$), and then read the expression aloud to the class: "The limit of x cubed as x approaches 0 from the left." Do the same for 1–2 additional examples that use a variety of limit notations (e.g., the symbol for infinity). Then have students pair up and take turns reading aloud different limit expressions to one another.
2	1.3 1.4	**Create Representations** Present students with a limit expression in analytical form (e.g., $\lim\limits_{x\to 3} \frac{x^2-9}{x-3}$), and then have them translate that expression into a variety of representations: constructing a graph, creating a table of values, and writing it as a verbal expression. Then have students check their graphs and tables using technology.
3	1.7	**Work Backward** Present students with a set of limit problems. Rather than determining the given limits, have them make a list of the various strategies that would be used to determine the limits (e.g., factoring, multiplying by conjugate, and simplify using trigonometric identities). After confirming their list is complete, have students work in pairs to create and write limit problems, each requiring one of the listed strategies. Then have them swap problems with another pair of students to complete each other's problems.
4	1.11	**Discussion Groups** Give each group of students a piecewise-defined function, a graph paper, and a list of x-values. Have them graph the function, then discuss whether the function is continuous or discontinuous at each x-value, and explain why. Ask students to take turns recording the group's conclusion for each x-value. If continuous, have students discuss and show that all three continuity conditions are satisfied. If discontinuous, have students state which condition was not satisfied.
5	1.16	**Think Aloud** In small groups, have students discuss the Intermediate Value Theorem and share ideas about real-world applications (e.g., speed of your car and weight of your kitten). Have groups post their ideas on a classroom wall using sticky notes.

TOPIC 1.1

Introducing Calculus: Can Change Occur at an Instant?

Required Course Content

ENDURING UNDERSTANDING

CHA-1

Calculus allows us to generalize knowledge about motion to diverse problems involving change.

LEARNING OBJECTIVE

CHA-1.A

Interpret the rate of change at an instant in terms of average rates of change over intervals containing that instant.

ESSENTIAL KNOWLEDGE

CHA-1.A.1

Calculus uses limits to understand and model dynamic change.

CHA-1.A.2

Because an average rate of change divides the change in one variable by the change in another, the average rate of change is undefined at a point where the change in the independent variable would be zero.

CHA-1.A.3

The limit concept allows us to define instantaneous rate of change in terms of average rates of change.

SUGGESTED SKILL

Connecting Representations

2.B

Identify mathematical information from graphical, numerical, analytical, and/or verbal representations.

SUGGESTED SKILL

Connecting Representations

2.B

Identify mathematical information from graphical, numerical, analytical, and/or verbal representations.

AVAILABLE RESOURCES

- Professional Development > **Definite Integrals: Interpreting Notational Expressions**
- AP Online Teacher Community Discussion > **How to "say" some of the notation**

TOPIC 1.2
Defining Limits and Using Limit Notation

Required Course Content

ENDURING UNDERSTANDING

LIM-1

Reasoning with definitions, theorems, and properties can be used to justify claims about limits.

LEARNING OBJECTIVE	ESSENTIAL KNOWLEDGE
LIM-1.A Represent limits analytically using correct notation.	**LIM-1.A.1** Given a function f, the limit of $f(x)$ as x approaches c is a real number R if $f(x)$ can be made arbitrarily close to R by taking x sufficiently close to c (but not equal to c). If the limit exists and is a real number, then the common notation is $\lim_{x \to c} f(x) = R$. **EXCLUSION STATEMENT** *The epsilon-delta definition of a limit is not assessed on the AP Calculus AB or BC Exam. However, teachers may include this topic in the course if time permits.*
LIM-1.B Interpret limits expressed in analytic notation.	**LIM-1.B.1** A limit can be expressed in multiple ways, including graphically, numerically, and analytically.

TOPIC 1.3

Estimating Limit Values from Graphs

Required Course Content

ENDURING UNDERSTANDING

LIM-1

Reasoning with definitions, theorems, and properties can be used to justify claims about limits.

LEARNING OBJECTIVE

LIM-1.C

Estimate limits of functions.

ESSENTIAL KNOWLEDGE

LIM-1.C.1

The concept of a limit includes one sided limits.

LIM-1.C.2

Graphical information about a function can be used to estimate limits.

LIM-1.C.3

Because of issues of scale, graphical representations of functions may miss important function behavior.

LIM-1.C.4

A limit might not exist for some functions at particular values of x. Some ways that the limit might not exist are if the function is unbounded, if the function is oscillating near this value, or if the limit from the left does not equal the limit from the right.

SUGGESTED SKILL

Connecting Representations

2.B

Identify mathematical information from graphical, numerical, analytical, and/or verbal representations.

ILLUSTRATIVE EXAMPLES

For LIM-1.C.4:

- $\lim\limits_{x\to 0}\dfrac{1}{x^2}=\infty$
- $\lim\limits_{x\to 0}\dfrac{|x|}{x}$ does not exist.
- $\lim\limits_{x\to 0}\sin\left(\dfrac{1}{x}\right)$ does not exist.
- $\lim\limits_{x\to 0}\dfrac{1}{x}$ does not exist.

AVAILABLE RESOURCES

- **AP Calculator Policy**
- Classroom Resource > **AP Calculus Use of Graphing Calculators**
- Professional Development > **Limits: Approximating Values and Functions**
- Classroom Resource > **Approximation**

SUGGESTED SKILL

Connecting Representations

2.B

Identify mathematical information from graphical, numerical, analytical, and/or verbal representations.

AVAILABLE RESOURCES

- **AP Calculator Policy**
- Classroom Resource > **AP Calculus Use of Graphing Calculators**
- Professional Development > **Limits: Approximating Values and Functions**
- Classroom Resource > **Approximation**

TOPIC 1.4

Estimating Limit Values from Tables

Required Course Content

ENDURING UNDERSTANDING

LIM-1

Reasoning with definitions, theorems, and properties can be used to justify claims about limits.

LEARNING OBJECTIVE

LIM-1.C

Estimate limits of functions.

ESSENTIAL KNOWLEDGE

LIM-1.C.5

Numerical information can be used to estimate limits.

TOPIC 1.5

Determining Limits Using Algebraic Properties of Limits

Required Course Content

ENDURING UNDERSTANDING

LIM-1

Reasoning with definitions, theorems, and properties can be used to justify claims about limits.

LEARNING OBJECTIVE

LIM-1.D

Determine the limits of functions using limit theorems.

ESSENTIAL KNOWLEDGE

LIM-1.D.1

One-sided limits can be determined analytically or graphically.

LIM-1.D.2

Limits of sums, differences, products, quotients, and composite functions can be found using limit theorems.

SUGGESTED SKILL

Implementing Mathematical Processes

1.E

Apply appropriate mathematical rules or procedures, with and without technology.

SUGGESTED SKILL

Implementing Mathematical Processes

1.C

Identify an appropriate mathematical rule or procedure based on the classification of a given expression.

ILLUSTRATIVE EXAMPLES

- Factoring and dividing common factors of rational functions
- Multiplying by an expression involving the conjugate of a sum or difference in order to simplify functions involving radicals
- Using alternate forms of trigonometric functions

TOPIC 1.6

Determining Limits Using Algebraic Manipulation

Required Course Content

ENDURING UNDERSTANDING

LIM-1

Reasoning with definitions, theorems, and properties can be used to justify claims about limits.

LEARNING OBJECTIVE

LIM-1.E

Determine the limits of functions using equivalent expressions for the function or the squeeze theorem.

ESSENTIAL KNOWLEDGE

LIM-1.E.1

It may be necessary or helpful to rearrange expressions into equivalent forms before evaluating limits.

TOPIC 1.7

Selecting Procedures for Determining Limits

This topic is intended to focus on the skill of selecting an appropriate procedure for determining limits. Students should be given opportunities to practice when and how to apply all learning objectives relating to determining limits.

SUGGESTED SKILL

Implementing Mathematical Processes

1.C

Identify an appropriate mathematical rule or procedure based on the classification of a given expression.

AVAILABLE RESOURCE

- Professional Development > **Selecting Procedures for Derivatives**

SUGGESTED SKILL

Justification

3.C

Confirm whether hypotheses or conditions of a selected definition, theorem, or test have been satisfied.

ILLUSTRATIVE EXAMPLES

The squeeze theorem can be used to show $\lim_{x \to 0} \frac{\sin x}{x} = 1$ and $\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$.

AVAILABLE RESOURCE

- AP Online Teacher Community Discussion > **Limits Questions**

TOPIC 1.8

Determining Limits Using the Squeeze Theorem

Required Course Content

ENDURING UNDERSTANDING

LIM-1

Reasoning with definitions, theorems, and properties can be used to justify claims about limits.

LEARNING OBJECTIVE

LIM-1.E

Determine the limits of functions using equivalent expressions for the function or the squeeze theorem.

ESSENTIAL KNOWLEDGE

LIM-1.E.2

The limit of a function may be found by using the squeeze theorem.

TOPIC 1.9

Connecting Multiple Representations of Limits

This topic is intended to focus on connecting representations. Students should be given opportunities to practice when and how to apply all learning objectives relating to limits and translating mathematical information from a single representation or across multiple representations.

SUGGESTED SKILL

Connecting Representations

2.C

Identify a re-expression of mathematical information presented in a given representation.

AVAILABLE RESOURCES

- **AP Calculator Policy**
- Classroom Resource > **AP Calculus Use of Graphing Calculators**
- Professional Development > **Limits: Approximating Values and Functions**

SUGGESTED SKILL

Justification

3.B

Identify an appropriate mathematical definition, theorem, or test to apply.

AVAILABLE RESOURCES

- AP Calculator Policy
- Classroom Resource > AP Calculus Use of Graphing Calculators

TOPIC 1.10

Exploring Types of Discontinuities

Required Course Content

ENDURING UNDERSTANDING

LIM-2

Reasoning with definitions, theorems, and properties can be used to justify claims about continuity.

LEARNING OBJECTIVE

LIM-2.A

Justify conclusions about continuity at a point using the definition.

ESSENTIAL KNOWLEDGE

LIM-2.A.1

Types of discontinuities include removable discontinuities, jump discontinuities, and discontinuities due to vertical asymptotes.

TOPIC 1.11

Defining Continuity at a Point

SUGGESTED SKILL

Justification

3.C

Confirm whether hypotheses or conditions of a selected definition, theorem, or test have been satisfied.

AVAILABLE RESOURCE

- AP Online Teacher Community Discussion > **Video on Continuity**

Required Course Content

ENDURING UNDERSTANDING

LIM-2

Reasoning with definitions, theorems, and properties can be used to justify claims about continuity.

LEARNING OBJECTIVE

LIM-2.A

Justify conclusions about continuity at a point using the definition.

ESSENTIAL KNOWLEDGE

LIM-2.A.2

A function f is continuous at $x = c$ provided that $f(c)$ exists, $\lim_{x \to c} f(x)$ exists, and $\lim_{x \to c} f(x) = f(c)$.

SUGGESTED SKILL

Implementing Mathematical Processes

1.E

Apply appropriate mathematical rules or procedures, with and without technology.

AVAILABLE RESOURCE

- AP Online Teacher Community Discussion > **Video on Continuity**

TOPIC 1.12
Confirming Continuity over an Interval

Required Course Content

ENDURING UNDERSTANDING

LIM-2

Reasoning with definitions, theorems, and properties can be used to justify claims about continuity.

LEARNING OBJECTIVE

LIM-2.B

Determine intervals over which a function is continuous.

ESSENTIAL KNOWLEDGE

LIM-2.B.1

A function is continuous on an interval if the function is continuous at each point in the interval.

LIM-2.B.2

Polynomial, rational, power, exponential, logarithmic, and trigonometric functions are continuous on all points in their domains.

TOPIC 1.13

Removing Discontinuities

SUGGESTED SKILL

Implementing Mathematical Processes

1.E

Apply appropriate mathematical rules or procedures, with and without technology.

AVAILABLE RESOURCE

- The Exam > **2012 Exam, MCQ #9**

Required Course Content

ENDURING UNDERSTANDING

LIM-2

Reasoning with definitions, theorems, and properties can be used to justify claims about continuity.

LEARNING OBJECTIVE

LIM-2.C

Determine values of x or solve for parameters that make discontinuous functions continuous, if possible.

ESSENTIAL KNOWLEDGE

LIM-2.C.1

If the limit of a function exists at a discontinuity in its graph, then it is possible to remove the discontinuity by defining or redefining the value of the function at that point, so it equals the value of the limit of the function as x approaches that point.

LIM-2.C.2

In order for a piecewise-defined function to be continuous at a boundary to the partition of its domain, the value of the expression defining the function on one side of the boundary must equal the value of the expression defining the other side of the boundary, as well as the value of the function at the boundary.

SUGGESTED SKILL

Justification

3.D

Apply an appropriate mathematical definition, theorem, or test.

TOPIC 1.14

Connecting Infinite Limits and Vertical Asymptotes

Required Course Content

ENDURING UNDERSTANDING

LIM-2

Reasoning with definitions, theorems, and properties can be used to justify claims about continuity.

LEARNING OBJECTIVE

LIM-2.D

Interpret the behavior of functions using limits involving infinity.

ESSENTIAL KNOWLEDGE

LIM-2.D.1

The concept of a limit can be extended to include infinite limits.

LIM-2.D.2

Asymptotic and unbounded behavior of functions can be described and explained using limits.

TOPIC 1.15

Connecting Limits at Infinity and Horizontal Asymptotes

SUGGESTED SKILL

Connecting Representations

2.D

Identify how mathematical characteristics or properties of functions are related in different representations.

Required Course Content

ENDURING UNDERSTANDING

LIM-2

Reasoning with definitions, theorems, and properties can be used to justify claims about continuity.

LEARNING OBJECTIVE

LIM-2.D

Interpret the behavior of functions using limits involving infinity.

ESSENTIAL KNOWLEDGE

LIM-2.D.3

The concept of a limit can be extended to include limits at infinity.

LIM-2.D.4

Limits at infinity describe end behavior.

LIM-2.D.5

Relative magnitudes of functions and their rates of change can be compared using limits.

SUGGESTED SKILL

Justification

3.E

Provide reasons or rationales for solutions or conclusions.

AVAILABLE RESOURCES

- Professional Development > **Continuity and Differentiability: Establishing Conditions for Definitions and Theorems**
- Classroom Resource > **Why We Use Theorem in Calculus**

TOPIC 1.16

Working with the Intermediate Value Theorem (IVT)

Required Course Content

ENDURING UNDERSTANDING

FUN-1

Existence theorems allow us to draw conclusions about a function's behavior on an interval without precisely locating that behavior.

LEARNING OBJECTIVE

FUN-1.A

Explain the behavior of a function on an interval using the Intermediate Value Theorem.

ESSENTIAL KNOWLEDGE

FUN-1.A.1

If f is a continuous function on the closed interval $[a, b]$ and d is a number between $f(a)$ and $f(b)$, then the Intermediate Value Theorem guarantees that there is at least one number c between a and b, such that $f(c) = d$.

AP CALCULUS AB AND BC

UNIT 2

Differentiation: Definition and Fundamental Properties

AP EXAM WEIGHTING	10–12% AB	4–7% BC
CLASS PERIODS	~13–14 AB	~9–10 BC

Remember to go to **AP Classroom** to assign students the online **Personal Progress Check** for this unit.

Whether assigned as homework or completed in class, the **Personal Progress Check** provides each student with immediate feedback related to this unit's topics and skills.

Personal Progress Check 2

Multiple-choice: ~30 questions
Free-response: 3 questions (partial)

UNIT 2

AP EXAM WEIGHTING	10–12% AB	4–7% BC
CLASS PERIODS	~13–14 AB	~9–10 BC

Differentiation: Definition and Fundamental Properties

BIG IDEA 1
Change **CHA**
- How can a state determine the rate of change in high school graduates at a particular level of public investment in education (in graduates per dollar) based on a model for the number of graduates as a function of the state's education budget?

BIG IDEA 2
Limits **LIM**
- Why do mathematical properties and rules for simplifying and evaluating limits apply to differentiation?

BIG IDEA 3
Analysis of Functions **FUN**
- If you knew that the rate of change in high school graduates at a particular level of public investment in education (in graduates per dollar) was a positive number, what might that tell you about the number of graduates at that level of investment?

Developing Understanding

Derivatives allow us to determine instantaneous rates of change. To develop understanding of how the definition of the derivative applies limits to average rates of change, create opportunities for students to explore average rates of change over increasingly small intervals. Graphing calculator explorations of how various operations affect slopes of tangent lines help students to make sense of basic rules and properties of differentiation. Encourage students to apply the order of operations as they select differentiation rules. Developing differentiation skills will allow students to model realistic instantaneous rates of change in Unit 4 and to analyze graphs in Unit 5.

Building the Mathematical Practices

1.E 2.B 4.C

Mathematicians know that a solution will only be as good as the procedure used to find it and that the difference between being correct and incorrect can often be traced to an arithmetic or procedural error. In other words, mathematicians know that the details matter. Students often find it difficult to apply mathematical procedures—including the rules of differentiation—with precision and accuracy. For example, students may drop important notation, such as a parenthesis, or misapply the product rule by taking the derivative of each factor separately and then multiplying those together. The content of Unit 2 is a foundational entry point for practicing the skill of applying mathematical procedures and learning to self-correct before common mistakes occur.

This is also an opportunity to revisit and reinforce the practice of connecting representations, as students will be seeing derivatives presented in analytical, numerical, graphical, and verbal representations. Students can practice by extracting information about the original function, f, from a graphical representation of f'. This can help prevent misunderstandings when examining the graph of a derivative (such as misinterpreting it as the graph of the original function instead).

Preparing for the AP Exam

Students should practice presenting clear mathematical structures that connect their work with definitions or theorems. For example, when asked to estimate the slope of the line tangent to a curve at a given point based on information provided in a table of values, as in **2013 AP Exam Free-Response Question 3 Part A**, students must present a difference quotient:

$$C'(3.5) \approx \frac{C(4)-C(3)}{4-3} = \frac{12.8-11.2}{1}.$$

Failure to present this structure will cost students the point they might have earned, even with a correct numerical answer. Similarly, to evaluate the derivative of $f(x)=u(x)\cdot v(x)$ at $x=3$, students should show the product rule structure, as in $f'(3)=u(3)v'(3)+v(3)u'(3)$, and import values. Finally, students should present mathematical expressions evaluated on the calculator and use specified rounding procedures, typically rounding or truncating to three places after the decimal point. It is helpful to establish the habit of storing intermediate calculations in the calculator in order to avoid accumulation of rounding errors.

UNIT AT A GLANCE

Enduring Understanding	Topic	Suggested Skills	Class Periods ~13–14 CLASS PERIODS (AB) ~9–10 CLASS PERIODS (BC)
CHA-2	**2.1 Defining Average and Instantaneous Rates of Change at a Point**	2.B Identify mathematical information from graphical, numerical, analytical, and/or verbal representations.	
	2.2 Defining the Derivative of a Function and Using Derivative Notation	1.D Identify an appropriate mathematical rule or procedure based on the relationship between concepts (e.g., rate of change and accumulation) or processes (e.g., differentiation and its inverse process, anti-differentiation) to solve problems. 4.C Use appropriate mathematical symbols and notation (e.g., Represent a derivative using $f'(x)$, y', and $\frac{dy}{dx}$).	
	2.3 Estimating Derivatives of a Function at a Point	1.E Apply appropriate mathematical rules or procedures, with and without technology.	
FUN-2	**2.4 Connecting Differentiability and Continuity: Determining When Derivatives Do and Do Not Exist**	3.E Provide reasons or rationales for solutions and conclusions.	
FUN-3	**2.5 Applying the Power Rule**	1.E Apply appropriate mathematical rules or procedures, with and without technology.	
	2.6 Derivative Rules: Constant, Sum, Difference, and Constant Multiple	1.E Apply appropriate mathematical rules or procedures, with and without technology.	
FUN-3 LIM-3	**2.7 Derivatives of $\cos x$, $\sin x$, e^x, and $\ln x$**	1.E Apply appropriate mathematical rules or procedures, with and without technology.	

UNIT AT A GLANCE *(cont'd)*

Enduring Understanding	Topic	Suggested Skills	Class Periods ~13–14 CLASS PERIODS (AB) ~9–10 CLASS PERIODS (BC)
FUN-3	**2.8 The Product Rule**	1.E Apply appropriate mathematical rules or procedures, with and without technology.	
	2.9 The Quotient Rule	1.E Apply appropriate mathematical rules or procedures, with and without technology.	
	2.10 Finding the Derivatives of Tangent, Cotangent, Secant, and/or Cosecant Functions	1.D Identify an appropriate mathematical rule or procedure based on the relationship between concepts (e.g., rate of change and accumulation) or processes (e.g., differentiation and its inverse process, anti-differentiation) to solve problems.	
	Go to **AP Classroom** to assign the **Personal Progress Check** for Unit 2. Review the results in class to identify and address any student misunderstandings.		

SAMPLE INSTRUCTIONAL ACTIVITIES

The sample activities on this page are optional and are offered to provide possible ways to incorporate various instructional approaches into the classroom. Teachers do not need to use these activities or instructional approaches and are free to alter or edit them. The examples below were developed in partnership with teachers from the AP community to share ways that they approach teaching some of the topics in this unit. Please refer to the Instructional Approaches section beginning on p. 199 for more examples of activities and strategies.

Activity	Topic	Sample Activity
1	2.1 2.2 2.3	**Graph and Switch** Present students with two or three functions and the graph of each function. Have each student choose a random derivative question and one function. Questions could include: Find the average rate of change on an interval, instantaneous rate of change at a point, derivative as a function, derivative value at a point, or equations for tangent or normal lines at a point. Have students answer their question and place their answer onto the function's graph. Then have students share their solutions with each other to give and receive feedback.
2	2.4	**Match Mine** Create cards containing graph images of functions with various continuous, discontinuous, differentiable, and nondifferentiable points or intervals. Provide each student in a pair with the same nine cards. Student A arranges their graphs in a 3×3 grid, which is not visible to Student B. Student A describes each of their graph's positions using information about continuity and differentiability to describe the graph. Based on the descriptions, Student B attempts to arrange their cards to match the grid of Student A.
3	2.5 2.6 2.7 2.8 2.9 2.10	**Error Analysis** Assign a function to each student. Ask them to find the function's derivative using one or more derivative rules. Allow them to check their answers. Ask half of the class to redo their work to include an error, thus having the wrong answer. Ask students to record their correct or incorrect work on a card. Mix up the cards and redistribute, having students determine if the answer is correct or incorrect. If incorrect, they should explain what error was made, and find the correct answer.
4	2.5 2.6 2.7 2.8 2.9 2.10	**Graphic Organizer** Provide students with colored paper, pens, and markers. Ask them to create a chart, a foldable card, or other creative method to organize all the derivative rules. For each rule, have them include the mathematical definition, examples, pictures, and helpful hints to understand and remember the rule.
5	2.8 2.9	**Round Table** Provide each student with the same worksheet containing four functions that require the product rule or quotient rule when finding the derivative. Then have students sit in groups of four. Each student determines the derivative of function No. 1, and then they pass their papers clockwise to the next student. Each student checks the first problem and, if necessary, discusses any mistakes with the previous student. Each student now completes function No. 2 on the paper, and the process continues until each student has their original paper back.

TOPIC 2.1

Defining Average and Instantaneous Rates of Change at a Point

SUGGESTED SKILL

Connecting Representations

2.B

Identify mathematical information from graphical, numerical, analytical, and/or verbal representations.

Required Course Content

ENDURING UNDERSTANDING

CHA-2

Derivatives allow us to determine rates of change at an instant by applying limits to knowledge about rates of change over intervals.

LEARNING OBJECTIVE

CHA-2.A

Determine average rates of change using difference quotients.

ESSENTIAL KNOWLEDGE

CHA-2.A.1

The difference quotients $\frac{f(a+h)-f(a)}{h}$ and $\frac{f(x)-f(a)}{x-a}$ express the average rate of change of a function over an interval.

CHA-2.B

Represent the derivative of a function as the limit of a difference quotient.

CHA-2.B.1

The instantaneous rate of change of a function at $x=a$ can be expressed by $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$ or $\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$, provided the limit exists. These are equivalent forms of the definition of the derivative and are denoted $f'(a)$.

SUGGESTED SKILLS

Implementing Mathematical Processes

1.D
Identify an appropriate mathematical rule or procedure based on the relationship between concepts or processes to solve problems.

Communication and Notation

4.C
Use appropriate mathematical symbols and notation.

AVAILABLE RESOURCES

- Professional Development > **Definite Integrals: Interpreting Notational Expressions**
- AP Online Teacher Community Discussion > **How to "Say" Some of the Notation**

TOPIC 2.2

Defining the Derivative of a Function and Using Derivative Notation

Required Course Content

ENDURING UNDERSTANDING

CHA-2
Derivatives allow us to determine rates of change at an instant by applying limits to knowledge about rates of change over intervals.

LEARNING OBJECTIVE

CHA-2.B
Represent the derivative of a function as the limit of a difference quotient.

ESSENTIAL KNOWLEDGE

CHA-2.B.2
The derivative of f is the function whose value at x is $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$, provided this limit exists.

CHA-2.B.3
For $y = f(x)$, notations for the derivative include $\frac{dy}{dx}$, $f'(x)$, and y'.

CHA-2.B.4
The derivative can be represented graphically, numerically, analytically, and verbally.

LEARNING OBJECTIVE

CHA-2.C
Determine the equation of a line tangent to a curve at a given point.

ESSENTIAL KNOWLEDGE

CHA-2.C.1
The derivative of a function at a point is the slope of the line tangent to a graph of the function at that point.

TOPIC 2.3

Estimating Derivatives of a Function at a Point

SUGGESTED SKILL

Implementing Mathematical Processes

1.E

Apply appropriate mathematical rules or procedures, with and without technology.

AVAILABLE RESOURCES

- Classroom Resource > **Approximation**
- Classroom Resource > **Reasoning from Tabular Data**

Required Course Content

ENDURING UNDERSTANDING

CHA-2

Derivatives allow us to determine rates of change at an instant by applying limits to knowledge about rates of change over intervals.

LEARNING OBJECTIVE

CHA-2.D

Estimate derivatives.

ESSENTIAL KNOWLEDGE

CHA-2.D.1

The derivative at a point can be estimated from information given in tables or graphs.

CHA-2.D.2

Technology can be used to calculate or estimate the value of a derivative of a function at a point.

SUGGESTED SKILL

Justification

3.E

Provide reasons or rationales for solutions and conclusions.

ILLUSTRATIVE EXAMPLES

For FUN-2.A.2:

- The left hand and right hand limits of the difference quotient are not equal, as in $f(x)=|x|$ at $x=0$.
- The tangent line is vertical and has no slope, as in $f(x)=\sqrt[3]{x}$ at $x=0$.

TOPIC 2.4
Connecting Differentiability and Continuity: Determining When Derivatives Do and Do Not Exist

Required Course Content

ENDURING UNDERSTANDING

FUN-2

Recognizing that a function's derivative may also be a function allows us to develop knowledge about the related behaviors of both.

LEARNING OBJECTIVE

FUN-2.A

Explain the relationship between differentiability and continuity.

ESSENTIAL KNOWLEDGE

FUN-2.A.1

If a function is differentiable at a point, then it is continuous at that point. In particular, if a point is not in the domain of f, then it is not in the domain of f'.

FUN-2.A.2

A continuous function may fail to be differentiable at a point in its domain.

TOPIC 2.5

Applying the Power Rule

SUGGESTED SKILL

Implementing Mathematical Processes

1.E

Apply appropriate mathematical rules or procedures, with and without technology.

AVAILABLE RESOURCE

- Professional Development > **Selecting Procedures for Derivatives**

Required Course Content

ENDURING UNDERSTANDING

FUN-3

Recognizing opportunities to apply derivative rules can simplify differentiation.

LEARNING OBJECTIVE

FUN-3.A

Calculate derivatives of familiar functions.

ESSENTIAL KNOWLEDGE

FUN-3.A.1

Direct application of the definition of the derivative and specific rules can be used to calculate the derivative for functions of the form $f(x) = x^r$.

SUGGESTED SKILL

Implementing Mathematical Processes

1.E

Apply appropriate mathematical rules or procedures, with and without technology.

AVAILABLE RESOURCE

- Professional Development > **Selecting Procedures for Derivatives**

TOPIC 2.6

Derivative Rules: Constant, Sum, Difference, and Constant Multiple

Required Course Content

ENDURING UNDERSTANDING

FUN-3

Recognizing opportunities to apply derivative rules can simplify differentiation.

LEARNING OBJECTIVE

FUN-3.A

Calculate derivatives of familiar functions.

ESSENTIAL KNOWLEDGE

FUN-3.A.2

Sums, differences, and constant multiples of functions can be differentiated using derivative rules.

FUN-3.A.3

The power rule combined with sum, difference, and constant multiple properties can be used to find the derivatives for polynomial functions.

TOPIC 2.7

Derivatives of cos x, sin x, e^x, and ln x

SUGGESTED SKILL

Implementing Mathematical Processes

1.E

Apply appropriate mathematical rules or procedures, with and without technology.

Required Course Content

ENDURING UNDERSTANDING

FUN-3

Recognizing opportunities to apply derivative rules can simplify differentiation.

LEARNING OBJECTIVE

FUN-3.A

Calculate derivatives of familiar functions.

ESSENTIAL KNOWLEDGE

FUN-3.A.4

Specific rules can be used to find the derivatives for sine, cosine, exponential, and logarithmic functions.

ENDURING UNDERSTANDING

LIM-3

Reasoning with definitions, theorems, and properties can be used to determine a limit.

LEARNING OBJECTIVE

LIM-3.A

Interpret a limit as a definition of a derivative.

ESSENTIAL KNOWLEDGE

LIM-3.A.1

In some cases, recognizing an expression for the definition of the derivative of a function whose derivative is known offers a strategy for determining a limit.

SUGGESTED SKILL

Implementing Mathematical Processes

1.E

Apply appropriate mathematical rules or procedures, with and without technology.

AVAILABLE RESOURCE

- Professional Development > **Selecting Procedures for Derivatives**

TOPIC 2.8

The Product Rule

Required Course Content

ENDURING UNDERSTANDING

FUN-3

Recognizing opportunities to apply derivative rules can simplify differentiation.

LEARNING OBJECTIVE

FUN-3.B

Calculate derivatives of products and quotients of differentiable functions.

ESSENTIAL KNOWLEDGE

FUN-3.B.1

Derivatives of products of differentiable functions can be found using the product rule.

TOPIC 2.9

The Quotient Rule

SUGGESTED SKILL

Implementing Mathematical Processes

1.E

Apply appropriate mathematical rules or procedures, with and without technology.

AVAILABLE RESOURCES

- Professional Development > **Selecting Procedures for Derivatives**
- AP Online Teacher Community Discussion > **Simplifying the Quotient Rule**

Required Course Content

ENDURING UNDERSTANDING

FUN-3

Recognizing opportunities to apply derivative rules can simplify differentiation.

LEARNING OBJECTIVE

FUN-3.B

Calculate derivatives of products and quotients of differentiable functions.

ESSENTIAL KNOWLEDGE

FUN-3.B.2

Derivatives of quotients of differentiable functions can be found using the quotient rule.

SUGGESTED SKILL

Implementing Mathematical Processes

1.D

Identify an appropriate mathematical rule or procedure based on the relationship between concepts or processes to solve problems.

TOPIC 2.10

Finding the Derivatives of Tangent, Cotangent, Secant, and/or Cosecant Functions

Required Course Content

ENDURING UNDERSTANDING

FUN-3

Recognizing opportunities to apply derivative rules can simplify differentiation.

LEARNING OBJECTIVE

FUN-3.B

Calculate derivatives of products and quotients of differentiable functions.

ESSENTIAL KNOWLEDGE

FUN-3.B.3

Rearranging tangent, cotangent, secant, and cosecant functions using identities allows differentiation using derivative rules.

AP CALCULUS AB AND BC

UNIT 3

Differentiation: Composite, Implicit, and Inverse Functions

AP EXAM WEIGHTING	9–13% AB	4–7% BC
CLASS PERIODS	~10–11 AB	~8–9 BC

Remember to go to **AP Classroom** to assign students the online **Personal Progress Check** for this unit.

Whether assigned as homework or completed in class, the **Personal Progress Check** provides each student with immediate feedback related to this unit's topics and skills.

Personal Progress Check 3

Multiple-choice: ~15 questions
Free-response: 3 questions (partial/full)

UNIT 3

AP EXAM WEIGHTING	9–13% AB	4–7% BC
CLASS PERIODS	~10–11 AB	~8–9 BC

Differentiation: Composite, Implicit, and Inverse Functions

BIG IDEA 3
Analysis of Functions **FUN**

- If pressure experienced by a diver is a function of depth and depth is a function of time, how might we find the rate of change in pressure with respect to time?

Developing Understanding

In this unit, students learn how to differentiate composite functions using the chain rule and apply that understanding to determine derivatives of implicit and inverse functions. Students need to understand that for composite functions, y is a function of u while u is a function of x. Leibniz notation for the chain rule, $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$, accounts for these relationships. Units analysis can strengthen the connection, as in $\frac{\text{psi}}{\text{min}} = \frac{\text{psi}}{\text{m}} \cdot \frac{\text{m}}{\text{min}}$. Saying, "times the derivative of what's inside," every time we apply the chain rule reminds students to avoid a common error. Mastering the chain rule is essential to success in all future units.

Building the Mathematical Practices

1.C 1.E 3.G

Identifying composite and implicit functions is a key differentiation skill. Students must recognize functions embedded in functions and be able to decompose composite functions into their "outer" and "inner" component functions. Misapplying the chain rule by forgetting to also differentiate the "inner" function or misidentifying the "inner" function are common errors. Provide sample responses that demonstrate these errors to help students be mindful of them in their own work. Reinforcing the chain rule structure sets the stage for Unit 6, when students learn the inverse of this process.

Students should continue to practice using correct notation and applying procedures accurately. Checking one another's work, reviewing sample responses (with and without errors), and using technology to check calculations develop these skills. Emphasize that taking higher-order derivatives mirrors familiar differentiation processes (i.e., "function is to first derivative as first derivative is to second derivative").

Use questioning techniques such as, "What does this mean?" to help students develop a more solid conceptual understanding of higher-order differentiation.

Preparing for the AP Exam

Mastery of the chain rule and its applications is essential for success on the AP Exam. The chain rule will be the target of assessment for many questions and a necessary step along the way for others. One common error is not recognizing when the chain rule applies, especially in composite functions such as $\sin^2 x$, $\tan(2x-1)$, and e^{x^2}. In expressions like $\frac{y}{3y^2-x}$, students must recognize that the chain rule applies to y because y depends on x. When multiple rules apply, students may struggle with the order of operations. Offer mixed practice differentiating general functions using select values provided in tables and graphs. Focus on products, quotients, compositions, and inverses of functions, especially those with names other than f and g. Connecting graphs, tables, and algebraic reasoning builds understanding of differentiation of inverse functions.

UNIT AT A GLANCE

Enduring Understanding	Topic	Suggested Skills	Class Periods ~10–11 CLASS PERIODS (AB) ~8–9 CLASS PERIODS (BC)
FUN-3	**3.1 The Chain Rule**	1.C Identify an appropriate mathematical rule or procedure based on the classification of a given expression (e.g., Use the chain rule to find the derivative of a composite function).	
	3.2 Implicit Differentiation	1.E Apply appropriate mathematical rules or procedures, with and without technology.	
	3.3 Differentiating Inverse Functions	3.G Confirm that solutions are accurate and appropriate.	
	3.4 Differentiating Inverse Trigonometric Functions	1.E Apply appropriate mathematical rules or procedures, with and without technology.	
	3.5 Selecting Procedures for Calculating Derivatives	1.C Identify an appropriate mathematical rule or procedure based on the classification of a given expression (e.g., Use the chain rule to find the derivative of a composite function).	
	3.6 Calculating Higher-Order Derivatives	1.E Apply appropriate mathematical rules or procedures, with and without technology.	

Go to **AP Classroom** to assign the **Personal Progress Check** for Unit 3. Review the results in class to identify and address any student misunderstandings.

SAMPLE INSTRUCTIONAL ACTIVITIES

The sample activities on this page are optional and are offered to provide possible ways to incorporate various instructional approaches into the classroom. Teachers do not need to use these activities or instructional approaches and are free to alter or edit them. The examples below were developed in partnership with teachers from the AP community to share ways that they approach teaching some of the topics in this unit. Please refer to the Instructional Approaches section beginning on p. 199 for more examples of activities and strategies.

Activity	Topic	Sample Activity
1	3.1	**Scavenger Hunt** Place a card with a starter question somewhere in the classroom, for example, "Find the derivative of $f(x) = \sin(4x)$." Place another card in the room with the solution to that card plus another question: "Solution: $4\cos(4x)$. Next problem: Find the derivative of $f(x) = (\sin(x))^4$." Continue posting solution cards with new problems until the final card presents a problem whose solution is on the original starter card (note that this solution should be added to the starter card above).
2	3.1	**Work Backward** Provide a chain rule problem with several possible answers. Have students identify which piece of the original problem contributed its derivative to one of the factors in the answer they are looking at. For each possible answer, students should say whether the answer is correct. They should also identify which piece of the composition was differentiated incorrectly or skipped, or state which factor of that particular answer did not come from the original problem. Students often have trouble knowing when to stop differentiating when learning e^x or trigonometric functions, so make sure you include these types of problems.
3	3.2	**Round Table** In groups of four, each student has an identical paper with four different problems on it. Students complete the first problem on their paper and then pass the paper clockwise to another member in their group. That student checks the first problem and then completes the second problem on the paper. Students rotate again and the process continues until each student has their own paper back.
4	3.4	**Quiz-Quiz-Trade** Give students a card containing a question and have them write the answer on the back. Students then circulate around the room and find a partner. One student quizzes the other by showing only the side of the card with the question on it, and then they reverse roles. They swap cards, find a new partner, quiz each other, and the process continues.

SUGGESTED SKILL

Implementing Mathematical Processes

1.C

Identify an appropriate mathematical rule or procedure based on the classification of a given expression.

AVAILABLE RESOURCE

- Professional Development > **Selecting Procedures for Derivatives**

TOPIC 3.1
The Chain Rule

Required Course Content

ENDURING UNDERSTANDING

FUN-3

Recognizing opportunities to apply derivative rules can simplify differentiation.

LEARNING OBJECTIVE

FUN-3.C

Calculate derivatives of compositions of differentiable functions.

ESSENTIAL KNOWLEDGE

FUN-3.C.1

The chain rule provides a way to differentiate composite functions.

TOPIC 3.2
Implicit Differentiation

SUGGESTED SKILL

Implementing Mathematical Processes

1.E

Apply appropriate mathematical rules or procedures, with and without technology.

Required Course Content

ENDURING UNDERSTANDING

FUN-3

Recognizing opportunities to apply derivative rules can simplify differentiation.

LEARNING OBJECTIVE

FUN-3.D

Calculate derivatives of implicitly defined functions.

ESSENTIAL KNOWLEDGE

FUN-3.D.1

The chain rule is the basis for implicit differentiation.

SUGGESTED SKILL

Justification

3.G

Confirm that solutions are accurate and appropriate.

TOPIC 3.3

Differentiating Inverse Functions

Required Course Content

ENDURING UNDERSTANDING

FUN-3

Recognizing opportunities to apply derivative rules can simplify differentiation.

LEARNING OBJECTIVE

FUN-3.E

Calculate derivatives of inverse and inverse trigonometric functions.

ESSENTIAL KNOWLEDGE

FUN-3.E.1

The chain rule and definition of an inverse function can be used to find the derivative of an inverse function, provided the derivative exists.

TOPIC 3.4

Differentiating Inverse Trigonometric Functions

SUGGESTED SKILL

Implementing Mathematical Processes

1.E

Apply appropriate mathematical rules or procedures, with and without technology.

Required Course Content

ENDURING UNDERSTANDING

FUN-3

Recognizing opportunities to apply derivative rules can simplify differentiation.

LEARNING OBJECTIVE

FUN-3.E

Calculate derivatives of inverse and inverse trigonometric functions.

ESSENTIAL KNOWLEDGE

FUN-3.E.2

The chain rule applied with the definition of an inverse function, or the formula for the derivative of an inverse function, can be used to find the derivatives of inverse trigonometric functions.

SUGGESTED SKILL

Implementing Mathematical Processes

1.C

Identify an appropriate mathematical rule or procedure based on the classification of a given expression.

AVAILABLE RESOURCE

- Professional Development > **Selecting Procedures for Derivatives**

TOPIC 3.5

Selecting Procedures for Calculating Derivatives

This topic is intended to focus on the skill of selecting an appropriate procedure for calculating derivatives. Students should be given opportunities to practice when and how to apply all learning objectives relating to calculating derivatives.

TOPIC 3.6

Calculating Higher - Order Derivatives

SUGGESTED SKILL

Implementing Mathematical Processes

1.E

Apply appropriate mathematical rules or procedures, with and without technology.

Required Course Content

ENDURING UNDERSTANDING

FUN-3

Recognizing opportunities to apply derivative rules can simplify differentiation.

LEARNING OBJECTIVE

FUN-3.F

Determine higher order derivatives of a function.

ESSENTIAL KNOWLEDGE

FUN-3.F.1

Differentiating f' produces the second derivative f'', provided the derivative of f' exists; repeating this process produces higher-order derivatives of f.

FUN-3.F.2

Higher-order derivatives are represented with a variety of notations. For $y = f(x)$, notations for the second derivative include $\frac{d^2y}{dx^2}$, $f''(x)$, and y''. Higher-order derivatives can be denoted $\frac{d^ny}{dx^n}$ or $f^{(n)}(x)$.

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AP CALCULUS AB AND BC

UNIT 4

Contextual Applications of Differentiation

AP EXAM WEIGHTING	10–15% AB	6–9% BC
CLASS PERIODS	~10–11 AB	~6–7 BC

Remember to go to **AP Classroom** to assign students the online **Personal Progress Check** for this unit.

Whether assigned as homework or completed in class, the **Personal Progress Check** provides each student with immediate feedback related to this unit's topics and skills.

Personal Progress Check 4

Multiple-choice: ~15 questions
Free-response: 3 questions

UNIT 4

AP EXAM WEIGHTING	10–15% AB	6–9% BC
CLASS PERIODS	~10–11 AB	~6–7 BC

Contextual Applications of Differentiation

BIG IDEA 1
Change **CHA**

- How are problems about position, velocity, and acceleration of a particle in motion over time structurally similar to problems about the volume of a rising balloon over an interval of heights, the population of London over the 14th century, or the metabolism of a dose of medicine over time?

BIG IDEA 2
Limits **LIM**

- Since certain indeterminate forms seem to actually approach a limit, how can we determine that limit, provided it exists?

Developing Understanding

Unit 4 begins by developing understanding of average and instantaneous rates of change in problems involving motion. The unit then identifies differentiation as a common underlying structure on which to build understanding of change in a variety of contexts. Students' understanding of units of measure often reinforces their understanding of contextual applications of differentiation. In problems involving related rates, identifying the independent variable common to related functions may help students to correctly apply the chain rule. When applying differentiation to determine limits of certain indeterminate forms using L'Hospital's rule, students must show that the rule applies.

Building the Mathematical Practices

1.D 1.F 2.A 3.F

Students will begin applying concepts from Units 2 and 3 to scenarios encountered in the world. Students often struggle to translate these verbal scenarios into the mathematical procedures necessary to answer the question. To solve these problems, students will need explicit instruction and intentional practice identifying key information, determining which procedure applies to the scenario presented (i.e., that "rates of change" indicate differentiation), stating what is changing and how, using correct units, and explaining what their answer means in the context of the scenario. Provide scenarios with different contexts but similar procedures so students begin to recognize and apply the reasoning behind those problem-solving decisions, rather than grasping at rules haphazardly.

Students must also be able to explain how an approximated value relates to the value it's intended to approximate. Students may not understand why they would use a tangent line approximation (i.e., linearization) rather than simply evaluating a function. Expose them to scenarios where an exact function value can't be calculated, and then ask them to determine whether a particular approximation is an overestimate or an underestimate of the function.

Preparing for the AP Exam

With contextual problems, emphasize careful reading for language such as, "find the rate of change." This will help students understand the underlying structure of the problem, answer the question asked, and interpret solutions in context. Students should not use words like "velocity" when they mean the rate of change in income, for example, even though the underlying structure is the same.

Emphasize that students must verify that $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$ (or that both approach infinity) as a necessary first step before applying L'Hospital's Rule to determine $\lim_{x \to a} \frac{f(x)}{g(x)}$. Students should understand that $\frac{0}{0}$ or $\frac{\infty}{\infty}$ are appropriate labels for indeterminate forms but do not represent values in an equation. Therefore, it is incorrect to write $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{0}{0}$, for example. Note that $\lim_{x \to a} \frac{f(x)}{g(x)} \neq \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$ when $\lim_{x \to a} g(x) = 0$. Also emphasize that the conclusion of L'Hospital's rule features the ratio of the derivatives of the numerator and denominator, respectively, rather than the derivative of the ratio.

UNIT AT A GLANCE

Enduring Understanding	Topic	Suggested Skills	Class Periods ~10–11 CLASS PERIODS (AB) ~6–7 CLASS PERIODS (BC)
CHA-3	**4.1 Interpreting the Meaning of the Derivative in Context**	**1.D** Identify an appropriate mathematical rule or procedure based on the relationship between concepts (e.g., rate of change and accumulation) or processes (e.g., differentiation and its inverse process, anti-differentiation) to solve problems.	
	4.2 Straight-Line Motion: Connecting Position, Velocity, and Acceleration	**1.E** Apply appropriate mathematical rules or procedures, with and without technology.	
	4.3 Rates of Change in Applied Contexts Other Than Motion	**2.A** Identify common underlying structures in problems involving different contextual situations.	
	4.4 Introduction to Related Rates	**1.E** Apply appropriate mathematical rules or procedures, with and without technology.	
	4.5 Solving Related Rates Problems	**3.F** Explain the meaning of mathematical solutions in context.	
	4.6 Approximating Values of a Function Using Local Linearity and Linearization	**1.F** Explain how an approximated value relates to the actual value.	
LIM-4	**4.7 Using L'Hospital's Rule for Determining Limits of Indeterminate Forms**	**3.D** Apply an appropriate mathematical definition, theorem, or test.	

Go to **AP Classroom** to assign the **Personal Progress Check** for Unit 4. Review the results in class to identify and address any student misunderstandings.

SAMPLE INSTRUCTIONAL ACTIVITIES

The sample activities on this page are optional and are offered to provide possible ways to incorporate various instructional approaches into the classroom. Teachers do not need to use these activities or instructional approaches and are free to alter or edit them. The examples below were developed in partnership with teachers from the AP community to share ways that they approach teaching some of the topics in this unit. Please refer to the Instructional Approaches section beginning on p. 199 for more examples of activities and strategies.

Activity	Topic	Sample Activity
1	4.1	**Quickwrite** Divide students into groups and give each group a context (outdoors, in a supermarket, in biology, in the government, at home, etc.). Students then write for a few minutes, listing things that are changing in that particular context.
2	4.2	**Create Representations** Provide verbal descriptions of a roller coaster ride: at time 0, velocity is 0 but about to become positive; at time 2, velocity is positive and increasing; at time 5, velocity is 0 and decreasing, etc. Have students graph position (from start), velocity, acceleration, speed, and then draw arrows at each point depicting whether their body would lean forward, backward, or not at all.
3	4.4	**Marking the Text** Have students read through a problem and highlight/underline the given quantities and directions in a problem, stating whether that information always applies or applies only at an instant.
4	4.5	**Round Table** Give students different related rates problems and a paper divided into five sections, titled as following: ▪ Draw a picture ▪ Equation ▪ Derivative ▪ Specific information used ▪ Interpretation Students first draw a picture of the situation and pass the papers clockwise. Students then critique the work in the previous section, complete the next section, and pass the papers again until all sections are completed.
5	4.6	**Scavenger Hunt** A starter question is posted in the room, for example, "Approximate the value . . . ." Have students work through the problem to find the value and then look for that value at the top of another card posted in the room. Students then solve the problem on that card, for example, "Write the equation of the tangent line . . . " and look for that solution on a third card, etc. The solution to the last problem will be on the starter card.

SUGGESTED SKILL

Implementing Mathematical Processes

1.D

Identify an appropriate mathematical rule or procedure based on the relationship between concepts or processes to solve problems.

AVAILABLE RESOURCE

- Professional Development > **Interpreting Context for Definite Integrals**

TOPIC 4.1
Interpreting the Meaning of the Derivative in Context

Required Course Content

ENDURING UNDERSTANDING

CHA-3

Derivatives allow us to solve real-world problems involving rates of change.

LEARNING OBJECTIVE

CHA-3.A

Interpret the meaning of a derivative in context.

ESSENTIAL KNOWLEDGE

CHA-3.A.1

The derivative of a function can be interpreted as the instantaneous rate of change with respect to its independent variable.

CHA-3.A.2

The derivative can be used to express information about rates of change in applied contexts.

CHA-3.A.3

The unit for $f'(x)$ is the unit for f divided by the unit for x.

TOPIC 4.2

Straight-Line Motion: Connecting Position, Velocity, and Acceleration

Required Course Content

ENDURING UNDERSTANDING

CHA-3

Derivatives allow us to solve real-world problems involving rates of change.

LEARNING OBJECTIVE

CHA-3.B

Calculate rates of change in applied contexts.

ESSENTIAL KNOWLEDGE

CHA-3.B.1

The derivative can be used to solve rectilinear motion problems involving position, speed, velocity, and acceleration.

SUGGESTED SKILL

Implementing Mathematical Processes

1.E

Apply appropriate mathematical rules or procedures, with and without technology.

SUGGESTED SKILL

Connecting Representations

2.A

Identify common underlying structures in problems involving different contextual situations.

TOPIC 4.3

Rates of Change in Applied Contexts Other Than Motion

Required Course Content

ENDURING UNDERSTANDING

CHA-3

Derivatives allow us to solve real-world problems involving rates of change.

LEARNING OBJECTIVE

CHA-3.C

Interpret rates of change in applied contexts.

ESSENTIAL KNOWLEDGE

CHA-3.C.1

The derivative can be used to solve problems involving rates of change in applied contexts.

TOPIC 4.4

Introduction to Related Rates

SUGGESTED SKILL

Implementing Mathematical Processes

1.E

Apply appropriate mathematical rules or procedures, with and without technology.

AVAILABLE RESOURCES

- Professional Development > **Related Rates: Analyzing Problems in Context**
- AP Online Teacher Community Discussion > **Related Rates in FRQ**

Required Course Content

ENDURING UNDERSTANDING

CHA-3

Derivatives allow us to solve real-world problems involving rates of change.

LEARNING OBJECTIVE

CHA-3.D

Calculate related rates in applied contexts.

ESSENTIAL KNOWLEDGE

CHA-3.D.1

The chain rule is the basis for differentiating variables in a related rates problem with respect to the same independent variable.

CHA-3.D.2

Other differentiation rules, such as the product rule and the quotient rule, may also be necessary to differentiate all variables with respect to the same independent variable.

SUGGESTED SKILL

Justification

3.F

Explain the meaning of mathematical solutions in context.

AVAILABLE RESOURCES

- Professional Development > **Related Rates: Analyzing Problems in Context**
- AP Online Teacher Community Discussion > **Related Rates in FRQ**

TOPIC 4.5
Solving Related Rates Problems

Required Course Content

ENDURING UNDERSTANDING

CHA-3

Derivatives allow us to solve real-world problems involving rates of change.

LEARNING OBJECTIVE

CHA-3.E

Interpret related rates in applied contexts.

ESSENTIAL KNOWLEDGE

CHA-3.E.1

The derivative can be used to solve related rates problems; that is, finding a rate at which one quantity is changing by relating it to other quantities whose rates of change are known.

TOPIC 4.6

Approximating Values of a Function Using Local Linearity and Linearization

Required Course Content

ENDURING UNDERSTANDING

CHA-3

Derivatives allow us to solve real-world problems involving rates of change.

LEARNING OBJECTIVE

CHA-3.F

Approximate a value on a curve using the equation of a tangent line.

ESSENTIAL KNOWLEDGE

CHA-3.F.1

The tangent line is the graph of a locally linear approximation of the function near the point of tangency.

CHA-3.F.2

For a tangent line approximation, the function's behavior near the point of tangency may determine whether a tangent line value is an underestimate or an overestimate of the corresponding function value.

SUGGESTED SKILL

Implementing Mathematical Processes

1.F

Explain how an approximated value relates to the actual value.

SUGGESTED SKILL

Justification

3.D

Apply an appropriate mathematical definition, theorem, or test.

AVAILABLE RESOURCES

- AP Online Teacher Community Discussion > **L'Hospital's Rule**
- AP Online Teacher Community Discussion > **Possible Inconsistent Language**
- The Exam > **2018 Chief Reader Report, FRQ #5(d)**
- The Exam > **2018 Samples and Commentary, FRQ #5(d)**
- The Exam > **2018 Scoring Guidelines, FRQ #5(d)**

TOPIC 4.7

Using L'Hospital's Rule for Determining Limits of Indeterminate Forms

Required Course Content

ENDURING UNDERSTANDING

LIM-4

L'Hospital's Rule allows us to determine the limits of some indeterminate forms.

LEARNING OBJECTIVE

LIM-4.A

Determine limits of functions that result in indeterminate forms.

ESSENTIAL KNOWLEDGE

LIM-4.A.1

When the ratio of two functions tends to $\frac{0}{0}$ or $\frac{\infty}{\infty}$ in the limit, such forms are said to be indeterminate.

EXCLUSION STATEMENT

There are many other indeterminate forms, such as $\infty - \infty$, for example, but these will not be assessed on either the AP Calculus AB or BC Exam. However, teachers may include these topics, if time permits.

LIM-4.A.2

Limits of the indeterminate forms $\frac{0}{0}$ or $\frac{\infty}{\infty}$ may be evaluated using L'Hospital's Rule.

AP CALCULUS AB AND BC

UNIT 5

Analytical Applications of Differentiation

AP EXAM WEIGHTING	15–18% AB	8–11% BC
CLASS PERIODS	~15–16 AB	~10–11 BC

Remember to go to **AP Classroom** to assign students the online **Personal Progress Check** for this unit.

Whether assigned as homework or completed in class, the **Personal Progress Check** provides each student with immediate feedback related to this unit's topics and skills.

Personal Progress Check 5

Multiple-choice: ~35 questions
Free-response: 3 questions

UNIT 5

AP EXAM WEIGHTING	15–18% AB	8–11% BC
CLASS PERIODS	~15–16 AB	~10–11 BC

Analytical Applications of Differentiation

BIG IDEA 3
Analysis of Functions **FUN**

- How might the Mean Value Theorem be used to justify a conclusion that you were speeding at some point on a certain stretch of highway, even without knowing the exact time you were speeding?
- What additional information is included in a sound mathematical argument about optimization that a simple description of an equivalent answer lacks?

Developing Understanding

In this unit, the superficial details of contextual applications of differentiation are stripped away to focus on abstract structures and formal conclusions. Reasoning with definitions and theorems establishes that answers and conclusions are more than conjectures; they have been analytically determined. As when students showed supporting work for answers in previous units, students will learn to present justifications for their conclusions about the behavior of functions over certain intervals or the locations of extreme values or points of inflection. The unit concludes this study of differentiation by applying abstract reasoning skills to justify solutions for realistic optimization problems.

Building the Mathematical Practices

1.E 2.D 2.E 3.E

The underlying processes of finding critical points and extrema are the foundation for the justifications students will write in this unit. Students should use calculators to graph a function and its derivatives to explore the related features of these graphs and confirm the results of their calculations.

Students often struggle with misinterpreting the characteristics of the graph of a derivative as though they are characteristics of the original function. Or, they use nonspecific language that conflates different functions (e.g., "it" rather than "f"). To prevent ongoing misconceptions, hold students accountable for extreme precision by having them practice matching graphs of functions to their derivatives and requiring them to explain their reasons to a peer.

Students also tend to rely on insufficient evidence or descriptions in their justifications, stating, for example, that "the graph of f is increasing because it's going up." This happens especially when examining derivative graphs on a calculator. Model calculus-based justifications (i.e., reasoning based on analysis of a derivative) both in discussion and in writing. Give students repeated opportunities to practice writing and revising their own justifications based on teacher feedback and feedback from their peers.

Preparing for the AP Exam

Sound reasoning must be accompanied by clear communication on the AP Exam. It may be helpful for students to use the language in the question as a starting point. Suppose a question asks, "Does g have a relative minimum, a relative maximum, or neither at $x=10$? Justify your answer." A student who writes, "g has neither a relative maximum nor a relative minimum at $x=10$, because . . . ," has begun well. Similarly, given a graph of the derivative, f', of a function, f, it is safer and easier for students to make arguments about f based directly on the graph of the derivative, as in, "f is concave up on $a<x<b$ because the graph of f' is increasing on $a<x<b$." Students should always refer to f, f', and f'' by name, rather than by "it" or "the function," which may leave the reader unsure of their intended meaning.

UNIT AT A GLANCE

Enduring Understanding	Topic	Suggested Skills	Class Periods ~15–16 CLASS PERIODS (AB) ~10–11 CLASS PERIODS (BC)
FUN-1	5.1 Using the Mean Value Theorem	3.E Provide reasons or rationales for solutions and conclusions.	
	5.2 Extreme Value Theorem, Global Versus Local Extrema, and Critical Points	3.E Provide reasons or rationales for solutions and conclusions.	
FUN-4	5.3 Determining Intervals on Which a Function is Increasing or Decreasing	2.E Describe the relationships among different representations of functions and their derivatives.	
	5.4 Using the First Derivative Test to Determine Relative (Local) Extrema	3.D Apply an appropriate mathematical definition, theorem, or test.	
	5.5 Using the Candidates Test to Determine Absolute (Global) Extrema	1.E Apply appropriate mathematical rules or procedures, with and without technology.	
	5.6 Determining Concavity of Functions over Their Domains	2.E Describe the relationships among different representations of functions and their derivatives.	
	5.7 Using the Second Derivative Test to Determine Extrema	3.D Apply an appropriate mathematical definition, theorem, or test.	
	5.8 Sketching Graphs of Functions and Their Derivatives	2.D Identify how mathematical characteristics or properties of functions are related in different representations.	
	5.9 Connecting a Function, Its First Derivative, and Its Second Derivative	2.D Identify how mathematical characteristics or properties of functions are related in different representations.	
	5.10 Introduction to Optimization Problems	2.A Identify common underlying structures in problems involving different contextual situations.	
	5.11 Solving Optimization Problems	3.F Explain the meaning of mathematical solutions in context.	
	5.12 Exploring Behaviors of Implicit Relations	1.E Apply appropriate mathematical rules or procedures, with and without technology. 3.E Provide reasons or rationales for solutions and conclusions.	

AP

Go to **AP Classroom** to assign the **Personal Progress Check** for Unit 5.
Review the results in class to identify and address any student misunderstandings.

SAMPLE INSTRUCTIONAL ACTIVITIES

The sample activities on this page are optional and are offered to provide possible ways to incorporate various instructional approaches into the classroom. Teachers do not need to use these activities or instructional approaches and are free to alter or edit them. The examples below were developed in partnership with teachers from the AP community to share ways that they approach teaching some of the topics in this unit. Please refer to the Instructional Approaches section beginning on p. 199 for more examples of activities and strategies.

Activity	Topic	Suggested Activity
1	5.3	**Critique Reasoning** Arrange students in groups of four to six, provide them with a function's derivative (e.g., $g'(x) = 5x + 3$), and ask them to determine if $g(x)$ is increasing or decreasing at a specific x-value, for example, $x = -3$. Ask students to share the reasoning for their conclusion with classmates in their group. Members of the group can then provide feedback and suggestions.
2	5.4 5.7	**Think-Pair-Share** Provide students with a graph of f' and a graph of f''. Ask them to identify relative extrema and practice writing justifications for relative extrema using the first or second derivative test. Once they've written their justification, ask them to pair with a partner and share their justifications. Students can then discuss similarities or differences in their justification wording.
3	5.5	**Create a Plan** Provide students with a function represented analytically on a closed interval. Ask them to discuss and write x-values that are viable candidates for absolute extrema. Once they have established the viable candidates, ask them to design a method for analyzing the behavior of the function's graph at the candidates and for identifying the extrema.
4	5.8 5.9	**Predict and Confirm** Provide students with the graph of a differentiable function, for example, $f(x) = x^3 - 4x^2 + 4x + 1$, but do not provide the rule for the function. Ask students to sketch a graph of the derivative of the function. Once students are done, reveal the rule for $f(x)$. Ask students to calculate $f'(x)$, and use technology to graph $f'(x)$ and compare it to their sketched graph.

SUGGESTED SKILL

Justification

3.E

Provide reasons or rationales for solutions and conclusions.

AVAILABLE RESOURCES

- Classroom Resource > **Why We Use Theorem in Calculus**
- AP Online Teacher Community Discussion > **Mean Value Existence Theorem**
- Professional Development > **Continuity and Differentiability: Establishing Conditions for Definitions and Theorems**

TOPIC 5.1
Using the Mean Value Theorem

Required Course Content

ENDURING UNDERSTANDING

FUN-1

Existence theorems allow us to draw conclusions about a function's behavior on an interval without precisely locating that behavior.

LEARNING OBJECTIVE

FUN-1.B

Justify conclusions about functions by applying the Mean Value Theorem over an interval.

ESSENTIAL KNOWLEDGE

FUN-1.B.1

If a function f is continuous over the interval $[a, b]$ and differentiable over the interval (a, b), then the Mean Value Theorem guarantees a point within that open interval where the instantaneous rate of change equals the average rate of change over the interval.

TOPIC 5.2

Extreme Value Theorem, Global Versus Local Extrema, and Critical Points

Required Course Content

ENDURING UNDERSTANDING

FUN-1

Existence theorems allow us to draw conclusions about a function's behavior on an interval without precisely locating that behavior.

LEARNING OBJECTIVE

FUN-1.C

Justify conclusions about functions by applying the Extreme Value Theorem.

ESSENTIAL KNOWLEDGE

FUN-1.C.1

If a function f is continuous over the interval $[a, b]$, then the Extreme Value Theorem guarantees that f has at least one minimum value and at least one maximum value on $[a, b]$.

FUN-1.C.2

A point on a function where the first derivative equals zero or fails to exist is a critical point of the function.

FUN-1.C.3

All local (relative) extrema occur at critical points of a function, though not all critical points are local extrema.

SUGGESTED SKILL

Justification

3.E

Provide reasons or rationales for solutions and conclusions.

AVAILABLE RESOURCES

- Classroom Resource > **Why We Use Theorem in Calculus**
- Professional Development > **Continuity and Differentiability: Establishing Conditions for Definitions and Theorems**
- Professional Development > **Justifying Properties and Behaviors of Functions**
- Classroom Resource > **Extrema**
- **On the Role of Sign Charts in AP Calculus Exams**

SUGGESTED SKILL

Connecting Representations

2.E

Describe the relationships among different representations of functions and their derivatives.

AVAILABLE RESOURCE

- The Exam > Commentary on the Instructions for the Free Response Section of the AP Calculus Exams
- On the Role of Sign Charts in AP Calculus Exams

TOPIC 5.3

Determining Intervals on Which a Function Is Increasing or Decreasing

Required Course Content

ENDURING UNDERSTANDING

FUN-4

A function's derivative can be used to understand some behaviors of the function.

LEARNING OBJECTIVE

FUN-4.A

Justify conclusions about the behavior of a function based on the behavior of its derivatives.

ESSENTIAL KNOWLEDGE

FUN-4.A.1

The first derivative of a function can provide information about the function and its graph, including intervals where the function is increasing or decreasing.

TOPIC 5.4

Using the First Derivative Test to Determine Relative (Local) Extrema

SUGGESTED SKILL

Justification

3.D

Apply an appropriate mathematical definition, theorem, or test.

AVAILABLE RESOURCE

- The Exam > **Commentary on the Instructions for the Free Response Section of the AP Calculus Exams**
- **On the Role of Sign Charts in AP Calculus Exams**

Required Course Content

ENDURING UNDERSTANDING

FUN-4

A function's derivative can be used to understand some behaviors of the function.

LEARNING OBJECTIVE

FUN-4.A

Justify conclusions about the behavior of a function based on the behavior of its derivatives.

ESSENTIAL KNOWLEDGE

FUN-4.A.2

The first derivative of a function can determine the location of relative (local) extrema of the function.

SUGGESTED SKILL

Implementing Mathematical Processes

1.E

Apply appropriate mathematical rules or procedures, with and without technology.

AVAILABLE RESOURCE

- The Exam > Commentary on the Instructions for the Free Response Section of the AP Calculus Exams
- On the Role of Sign Charts in AP Calculus Exams

TOPIC 5.5
Using the Candidates Test to Determine Absolute (Global) Extrema

Required Course Content

ENDURING UNDERSTANDING

FUN-4

A function's derivative can be used to understand some behaviors of the function.

LEARNING OBJECTIVE

FUN-4.A

Justify conclusions about the behavior of a function based on the behavior of its derivatives.

ESSENTIAL KNOWLEDGE

FUN-4.A.3

Absolute (global) extrema of a function on a closed interval can only occur at critical points or at endpoints.

TOPIC 5.6

Determining Concavity of Functions over Their Domains

Required Course Content

ENDURING UNDERSTANDING

FUN-4

A function's derivative can be used to understand some behaviors of the function.

LEARNING OBJECTIVE

FUN-4.A

Justify conclusions about the behavior of a function based on the behavior of its derivatives.

ESSENTIAL KNOWLEDGE

FUN-4.A.4

The graph of a function is concave up (down) on an open interval if the function's derivative is increasing (decreasing) on that interval.

FUN-4.A.5

The second derivative of a function provides information about the function and its graph, including intervals of upward or downward concavity.

FUN-4.A.6

The second derivative of a function may be used to locate points of inflection for the graph of the original function.

SUGGESTED SKILL

Connecting Representations

2.E

Describe the relationships among different representations of functions and their derivatives.

AVAILABLE RESOURCE

- AP Online Teacher Community Discussion > **Second Derivative Test Wording and Justifying Concavity Intervals**

SUGGESTED SKILL

Justification

3.D

Apply an appropriate mathematical definition, theorem, or test.

AVAILABLE RESOURCE

- The Exam > **Commentary on the Instructions for the Free Response Section of the AP Calculus Exams**
- **On the Role of Sign Charts in AP Calculus Exams**

TOPIC 5.7

Using the Second Derivative Test to Determine Extrema

Required Course Content

ENDURING UNDERSTANDING

FUN-4

A function's derivative can be used to understand some behaviors of the function.

LEARNING OBJECTIVE

FUN-4.A

Justify conclusions about the behavior of a function based on the behavior of its derivatives.

ESSENTIAL KNOWLEDGE

FUN-4.A.7

The second derivative of a function may determine whether a critical point is the location of a relative (local) maximum or minimum.

FUN-4.A.8

When a continuous function has only one critical point on an interval on its domain and the critical point corresponds to a relative (local) extremum of the function on the interval, then that critical point also corresponds to the absolute (global) extremum of the function on the interval.

TOPIC 5.8

Sketching Graphs of Functions and Their Derivatives

SUGGESTED SKILL

Connecting Representations

2.D

Identify how mathematical characteristics or properties of functions are related in different representations.

Required Course Content

ENDURING UNDERSTANDING

FUN-4

A function's derivative can be used to understand some behaviors of the function.

LEARNING OBJECTIVE

FUN-4.A

Justify conclusions about the behavior of a function based on the behavior of its derivatives.

ESSENTIAL KNOWLEDGE

FUN-4.A.9

Key features of functions and their derivatives can be identified and related to their graphical, numerical, and analytical representations.

FUN-4.A.10

Graphical, numerical, and analytical information from f' and f'' can be used to predict and explain the behavior of f.

SUGGESTED SKILL

Connecting Representations

2.D

Identify how mathematical characteristics or properties of functions are related in different representations.

AVAILABLE RESOURCE

- Professional Development > **Justifying Properties and Behaviors of Functions**

TOPIC 5.9
Connecting a Function, Its First Derivative, and Its Second Derivative

Required Course Content

ENDURING UNDERSTANDING

FUN-4

A function's derivative can be used to understand some behaviors of the function.

LEARNING OBJECTIVE

FUN-4.A

Justify conclusions about the behavior of a function based on the behavior of its derivatives.

ESSENTIAL KNOWLEDGE

FUN-4.A.11

Key features of the graphs of f, f', and f'' are related to one another.

TOPIC 5.10

Introduction to Optimization Problems

SUGGESTED SKILL

Connecting Representations

2.A

Identify common underlying structures in problems involving different contextual situations.

Required Course Content

ENDURING UNDERSTANDING

FUN-4

A function's derivative can be used to understand some behaviors of the function.

LEARNING OBJECTIVE

FUN-4.B

Calculate minimum and maximum values in applied contexts or analysis of functions.

ESSENTIAL KNOWLEDGE

FUN-4.B.1

The derivative can be used to solve optimization problems; that is, finding a minimum or maximum value of a function on a given interval.

SUGGESTED SKILL

Justification

3.F

Explain the meaning of mathematical solutions in context.

TOPIC 5.11

Solving Optimization Problems

Required Course Content

ENDURING UNDERSTANDING

FUN-4

A function's derivative can be used to understand some behaviors of the function.

LEARNING OBJECTIVE

FUN-4.C

Interpret minimum and maximum values calculated in applied contexts.

ESSENTIAL KNOWLEDGE

FUN-4.C.1

Minimum and maximum values of a function take on specific meanings in applied contexts.

TOPIC 5.12

Exploring Behaviors of Implicit Relations

SUGGESTED SKILLS

Implementing Mathematical Processes

1.E
Apply appropriate mathematical rules or procedures, with and without technology.

Justification

3.E
Provide reasons or rationales for solutions and conclusions.

Required Course Content

ENDURING UNDERSTANDING

FUN-4
A function's derivative can be used to understand some behaviors of the function.

LEARNING OBJECTIVE

FUN-4.D
Determine critical points of implicit relations.

ESSENTIAL KNOWLEDGE

FUN-4.D.1
A point on an implicit relation where the first derivative equals zero or does not exist is a critical point of the function.

LEARNING OBJECTIVE

FUN-4.E
Justify conclusions about the behavior of an implicitly defined function based on evidence from its derivatives.

ESSENTIAL KNOWLEDGE

FUN-4.E.1
Applications of derivatives can be extended to implicitly defined functions.

FUN-4.E.2
Second derivatives involving implicit differentiation may be relations of x, y, and $\frac{dy}{dx}$.

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AP CALCULUS AB AND BC

UNIT 6

Integration and Accumulation of Change

AP EXAM WEIGHTING	17–20% AB	17–20% BC
CLASS PERIODS	~18–20 AB	~15–16 BC

Remember to go to **AP Classroom** to assign students the online **Personal Progress Check** for this unit.

Whether assigned as homework or completed in class, the **Personal Progress Check** provides each student with immediate feedback related to this unit's topics and skills.

Personal Progress Check 6

Multiple-choice:
~25 questions (AB)
~35 questions (BC)
Free-response: 3 questions

UNIT 6

AP EXAM WEIGHTING	17–20% AB	17–20% BC
CLASS PERIODS	~18–20 AB	~15–16 BC

Integration and Accumulation of Change

BIG IDEA 1
Change CHA
- Given information about a rate of population growth over time, how can we determine how much the population changed over a given interval of time?

BIG IDEA 2
Limits LIM
- If compounding more often increases the amount in an account with a given rate of return and term, why doesn't compounding continuously result in an infinite account balance, all other things being equal?

BIG IDEA 3
Analysis of Functions FUN
- How is integrating to find areas related to differentiating to find slopes?

Developing Understanding

This unit establishes the relationship between differentiation and integration using the Fundamental Theorem of Calculus. Students begin by exploring the contextual meaning of areas of certain regions bounded by rate functions. Integration determines accumulation of change over an interval, just as differentiation determines instantaneous rate of change at a point. Students should understand that integration is a limiting case of a sum of products (areas) in the same way that differentiation is a limiting case of a quotient of differences (slopes). Future units will apply the idea of accumulation of change to a variety of realistic and geometric applications.

Building the Mathematical Practices

1.D 1.E 2.C 3.D

Students often struggle with the relationship between differentiation and integration. They think that integration is simply differentiation in reverse order. However, to apply the rules of integration correctly, students must think more strategically, taking into consideration how the "pieces" fit together. Students will need explicit guidance for choosing an appropriate antidifferentiation strategy that's based on the underlying patterns in different categories of integrands (e.g., using u-substitution when they recognize that the integrand is a factor of the derivative of a composite function or using integration by parts for an integrand, udv, that is related to a term in the derivative of the product uv **BC ONLY**).

Students also struggle with relating a symbolic limit of a Riemann sum to that limit expressed as a definite integral, because of the complexity of the expressions. To help students feel more comfortable working with these expressions, use explicit strategies, such as helping students to break complex expressions into familiar components, or matching expressions for a definite integral with the limit of a Riemann sum, and vice versa.

Preparing for the AP Exam

Students should be careful applying the chain rule, both when differentiating functions defined by integrals and when integrating using u-substitution. Students will need to recognize integrands that are factors of a chain rule derivative and should practice u-substitution until the process is internalized. Students will additionally need to recognize integrands that suggest strategies such as integration by parts or partial fractions and should use mixed practice in preparation for the exam **BC ONLY**.

When using a calculator to evaluate a definite integral in a free-response question, students should present the expression for the definite integral, including endpoints of integration, and an appropriately placed differential. When evaluating an integral without a calculator, students should present an appropriate antiderivative; they should include a constant of integration with indefinite integrals. As always, students should be careful about parentheses usage and should avoid writing strings of equal signs equating expressions that are not equal.

UNIT AT A GLANCE

Enduring Understanding	Topic	Suggested Skills	Class Periods ~18–20 CLASS PERIODS (AB) ~15–16 CLASS PERIODS (BC)
CHA-4	**6.1 Exploring Accumulations of Change**	4.B Use appropriate units of measure.	
LIM-5	**6.2 Approximating Areas with Riemann Sums**	1.F Explain how an approximated value relates to the actual value.	
	6.3 Riemann Sums, Summation Notation, and Definite Integral Notation	2.C Identify a re-expression of mathematical information presented in a given representation.	
FUN-5	**6.4 The Fundamental Theorem of Calculus and Accumulation Functions**	1.D Identify an appropriate mathematical rule or procedure based on the relationship between concepts (e.g., rate of change and accumulation) or processes (e.g., differentiation and its inverse process, anti-differentiation) to solve problems.	
	6.5 Interpreting the Behavior of Accumulation Functions Involving Area	2.D Identify how mathematical characteristics or properties of functions are related in different representations.	
FUN-6	**6.6 Applying Properties of Definite Integrals**	3.D Apply an appropriate mathematical definition, theorem, or test.	
	6.7 The Fundamental Theorem of Calculus and Definite Integrals	3.D Apply an appropriate mathematical definition, theorem, or test.	
	6.8 Finding Antiderivatives and Indefinite Integrals: Basic Rules and Notation	4.C Use appropriate mathematical symbols and notation (e.g., Represent a derivative using $f'(x)$, y', and $\frac{dy}{dx}$).	
	6.9 Integrating Using Substitution	1.E Apply appropriate mathematical rules or procedures, with and without technology.	
	6.10 Integrating Functions Using Long Division and Completing the Square	1.E Apply appropriate mathematical rules or procedures, with and without technology.	

continued on next page

UNIT AT A GLANCE *(cont'd)*

Enduring Understanding	Topic	Suggested Skills	Class Periods ~18–20 CLASS PERIODS (AB) ~15–16 CLASS PERIODS (BC)
FUN-6	**6.11 Integrating Using Integration by Parts** BC ONLY	1.E Apply appropriate mathematical rules or procedures, with and without technology.	
FUN-6	**6.12 Integrating Using Linear Partial Fractions** BC ONLY	1.E Apply appropriate mathematical rules or procedures, with and without technology.	
LIM-6	**6.13 Evaluating Improper Integrals** BC ONLY	1.E Apply appropriate mathematical rules or procedures, with and without technology.	
FUN-6	**6.14 Selecting Techniques for Antidifferentiation**	1.C Identify an appropriate mathematical rule or procedure based on the classification of a given expression (e.g., Use the chain rule to find the derivative of a composite function).	

Go to **AP Classroom** to assign the **Personal Progress Check** for Unit 6. Review the results in class to identify and address any student misunderstandings.

SAMPLE INSTRUCTIONAL ACTIVITIES

The sample activities on this page are optional and are offered to provide possible ways to incorporate various instructional approaches into the classroom. Teachers do not need to use these activities or instructional approaches and are free to alter or edit them. The examples below were developed in partnership with teachers from the AP community to share ways that they approach teaching some of the topics in this unit. Please refer to the Instructional Approaches section beginning on p. 199 for more examples of activities and strategies.

Activity	Topic	Sample Activity
1	6.3	**Quickwrite** Present the class with several examples of definite integrals set equal to Riemann sums in summation notation, for example $\int_{-2}^{5}(x^2+5)dx=\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{7}{n}\right)\left(\left(-2+\frac{7}{n}i\right)^2+5\right)$. Ask students to take five minutes to identify and write about all common elements between the two expressions and why they think the two expressions are equivalent. After finishing the five minutes, ask students to share their observations with the class.
2	6.9	**Look For a Pattern** Present students with several indefinite integrals and proposed, yet incorrect, antiderivatives, for example, $\int(5x+2)^{20}\,dx=\frac{1}{21}(5x+2)^{21}+C$. Ask them to check the antiderivatives by differentiating each and comparing to the original integrands. As students see that each antiderivative is incorrect, ask them to identify a pattern within the errors. Identifying this pattern will establish the foundation for integrating using substitution.
3	6.10	**Odd One Out** To help students select a strategy, form groups of four, presenting each student an indefinite integral whose integrand is rational. For each group, include one integrand that requires long division or completing the square. Ask students to decide if their example fits with the group. Identifying the odd one out will help students connect integrand form to the appropriate strategy.

TOPIC 6.1

Exploring Accumulations of Change

SUGGESTED SKILL

Communication and Notation

4.B

Use appropriate units of measure.

Required Course Content

ENDURING UNDERSTANDING

CHA-4

Definite integrals allow us to solve problems involving the accumulation of change over an interval.

LEARNING OBJECTIVE

CHA-4.A

Interpret the meaning of areas associated with the graph of a rate of change in context.

ESSENTIAL KNOWLEDGE

CHA-4.A.1

The area of the region between the graph of a rate of change function and the x axis gives the accumulation of change.

CHA-4.A.2

In some cases, accumulation of change can be evaluated by using geometry.

CHA-4.A.3

If a rate of change is positive (negative) over an interval, then the accumulated change is positive (negative).

CHA-4.A.4

The unit for the area of a region defined by rate of change is the unit for the rate of change multiplied by the unit for the independent variable.

SUGGESTED SKILL

Implementing Mathematical Processes

1.F

Explain how an approximated value relates to the actual value.

AVAILABLE RESOURCE

- Classroom Resource > **Reasoning from Tabular Data**

TOPIC 6.2

Approximating Areas with Riemann Sums

Required Course Content

ENDURING UNDERSTANDING

LIM-5

Definite integrals can be approximated using geometric and numerical methods.

LEARNING OBJECTIVE

LIM-5.A

Approximate a definite integral using geometric and numerical methods.

ESSENTIAL KNOWLEDGE

LIM-5.A.1

Definite integrals can be approximated for functions that are represented graphically, numerically, analytically, and verbally.

LIM-5.A.2

Definite integrals can be approximated using a left Riemann sum, a right Riemann sum, a midpoint Riemann sum, or a trapezoidal sum; approximations can be computed using either uniform or nonuniform partitions.

LIM-5.A.3

Definite integrals can be approximated using numerical methods, with or without technology.

LIM-5.A.4

Depending on the behavior of a function, it may be possible to determine whether an approximation for a definite integral is an underestimate or overestimate for the value of the definite integral.

TOPIC 6.3

Riemann Sums, Summation Notation, and Definite Integral Notation

Required Course Content

ENDURING UNDERSTANDING

LIM-5

Definite integrals can be approximated using geometric and numerical methods.

LEARNING OBJECTIVE

LIM-5.B

Interpret the limiting case of the Riemann sum as a definite integral.

ESSENTIAL KNOWLEDGE

LIM-5.B.1

The limit of an approximating Riemann sum can be interpreted as a definite integral.

LIM-5.B.2

A Riemann sum, which requires a partition of an interval I, is the sum of products, each of which is the value of the function at a point in a subinterval multiplied by the length of that subinterval of the partition.

LEARNING OBJECTIVE

LIM-5.C

Represent the limiting case of the Riemann sum as a definite integral.

ESSENTIAL KNOWLEDGE

LIM-5.C.1

The definite integral of a continuous function f over the interval $[a, b]$, denoted by $\int_a^b f(x)dx$, is the limit of Riemann sums as the widths of the subintervals approach 0. That is, $\int_a^b f(x)dx = \lim\limits_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$, where n is the number of subintervals, Δx_i is the width of the ith subinterval, and x_i^* is a value in the ith subinterval.

LIM-5.C.2

A definite integral can be translated into the limit of a related Riemann sum, and the limit of a Riemann sum can be written as a definite integral.

SUGGESTED SKILL

Connecting Representations

2.C

Identify a re-expression of mathematical information presented in a given representation.

AVAILABLE RESOURCES

- Professional Development > **Definite Integrals: Interpreting Notational Expressions**
- AP Online Teacher Community Discussion > **How to "Say" Some of the Notation**
- AP Online Teacher Community Discussion > **Definite Integral as the Limit of a Riemann Sum**

SUGGESTED SKILL

Implementing Mathematical Processes

1.D

Identify an appropriate mathematical rule or procedure based on the relationship between concepts or processes to solve problems.

ILLUSTRATIVE EXAMPLES

For FUN-5.A.1:

$f(x) = \int_0^x e^{-t^2}\,dt.$

AVAILABLE RESOURCES

- Professional Development > **The Fundamental Theorem of Calculus**
- Classroom Resource > **Functions Defined by Integrals**

TOPIC 6.4
The Fundamental Theorem of Calculus and Accumulation Functions

Required Course Content

ENDURING UNDERSTANDING

FUN-5

The Fundamental Theorem of Calculus connects differentiation and integration.

LEARNING OBJECTIVE

FUN-5.A

Represent accumulation functions using definite integrals.

ESSENTIAL KNOWLEDGE

FUN-5.A.1

The definite integral can be used to define new functions.

FUN-5.A.2

If f is a continuous function on an interval containing a, then $\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)$, where x is in the interval.

TOPIC 6.5

Interpreting the Behavior of Accumulation Functions Involving Area

Required Course Content

ENDURING UNDERSTANDING

FUN-5

The Fundamental Theorem of Calculus connects differentiation and integration.

LEARNING OBJECTIVE

FUN-5.A

Represent accumulation functions using definite integrals.

ESSENTIAL KNOWLEDGE

FUN-5.A.3

Graphical, numerical, analytical, and verbal representations of a function f provide information about the function g defined as $g(x) = \int_a^x f(t)dt$.

SUGGESTED SKILL

Connecting Representations

2.D

Identify how mathematical characteristics or properties of functions are related in different representations.

SUGGESTED SKILL

Justification

3.D

Apply an appropriate mathematical definition, theorem, or test.

TOPIC 6.6

Applying Properties of Definite Integrals

Required Course Content

ENDURING UNDERSTANDING

FUN-6

Recognizing opportunities to apply knowledge of geometry and mathematical rules can simplify integration.

LEARNING OBJECTIVE

FUN-6.A

Calculate a definite integral using areas and properties of definite integrals.

ESSENTIAL KNOWLEDGE

FUN-6.A.1

In some cases, a definite integral can be evaluated by using geometry and the connection between the definite integral and area.

FUN-6.A.2

Properties of definite integrals include the integral of a constant times a function, the integral of the sum of two functions, reversal of limits of integration, and the integral of a function over adjacent intervals.

FUN-6.A.3

The definition of the definite integral may be extended to functions with removable or jump discontinuities.

TOPIC 6.7

The Fundamental Theorem of Calculus and Definite Integrals

Required Course Content

ENDURING UNDERSTANDING

FUN-6

Recognizing opportunities to apply knowledge of geometry and mathematical rules can simplify integration.

LEARNING OBJECTIVE

FUN-6.B

Evaluate definite integrals analytically using the Fundamental Theorem of Calculus.

ESSENTIAL KNOWLEDGE

FUN-6.B.1

An antiderivative of a function f is a function g whose derivative is f.

FUN-6.B.2

If a function f is continuous on an interval containing a, the function defined by $F(x) = \int_a^x f(t)\,dt$ is an antiderivative of f for x in the interval.

FUN-6.B.3

If f is continuous on the interval $[a, b]$ and F is an antiderivative of f, then $\int_a^b f(x)\,dx = F(b) - F(a)$.

SUGGESTED SKILL

Justification

3.D

Apply an appropriate mathematical definition, theorem, or test.

AVAILABLE RESOURCE

- Professional Development > **The Fundamental Theorem of Calculus**

SUGGESTED SKILL

Communication and Notation

4.C

Use appropriate mathematical symbols and notation.

TOPIC 6.8

Finding Antiderivatives and Indefinite Integrals: Basic Rules and Notation

Required Course Content

ENDURING UNDERSTANDING

FUN-6

Recognizing opportunities to apply knowledge of geometry and mathematical rules can simplify integration.

LEARNING OBJECTIVE

FUN-6.C

Determine antiderivatives of functions and indefinite integrals, using knowledge of derivatives.

ESSENTIAL KNOWLEDGE

FUN-6.C.1

$\int f(x)dx$ is an indefinite integral of the function f and can be expressed as $\int f(x)dx = F(x) + C$, where $F'(x) = f(x)$ and C is any constant.

FUN-6.C.2

Differentiation rules provide the foundation for finding antiderivatives.

FUN-6.C.3

Many functions do not have closed-form antiderivatives.

TOPIC 6.9

Integrating Using Substitution

SUGGESTED SKILL

Implementing Mathematical Processes

1.E

Apply appropriate mathematical rules or procedures, with and without technology.

AVAILABLE RESOURCES

- Professional Development > **Applying Procedures for Integration by Substitution**
- AP Online Teacher Community Discussion > **U-Substitution with Improper Integrals**

Required Course Content

ENDURING UNDERSTANDING

FUN-6

Recognizing opportunities to apply knowledge of geometry and mathematical rules can simplify integration.

LEARNING OBJECTIVE

FUN-6.D

For integrands requiring substitution or rearrangements into equivalent forms:

(a) Determine indefinite integrals.

(b) Evaluate definite integrals.

ESSENTIAL KNOWLEDGE

FUN-6.D.1

Substitution of variables is a technique for finding antiderivatives.

FUN-6.D.2

For a definite integral, substitution of variables requires corresponding changes to the limits of integration.

SUGGESTED SKILL

Implementing Mathematical Processes

1.E

Apply appropriate mathematical rules or procedures, with and without technology.

TOPIC 6.10

Integrating Functions Using Long Division and Completing the Square

Required Course Content

ENDURING UNDERSTANDING

FUN-6

Recognizing opportunities to apply knowledge of geometry and mathematical rules can simplify integration.

LEARNING OBJECTIVE

FUN-6.D

For integrands requiring substitution or rearrangements into equivalent forms:

(a) Determine indefinite integrals.

(b) Evaluate definite integrals.

ESSENTIAL KNOWLEDGE

FUN-6.D.3

Techniques for finding antiderivatives include rearrangements into equivalent forms, such as long division and completing the square.

TOPIC 6.11

Integrating Using Integration by Parts BC ONLY

SUGGESTED SKILL

Implementing Mathematical Processes

1.E

Apply appropriate mathematical rules or procedures, with and without technology.

Required Course Content

ENDURING UNDERSTANDING

FUN-6

Recognizing opportunities to apply knowledge of geometry and mathematical rules can simplify integration.

LEARNING OBJECTIVE

FUN-6.E

For integrands requiring integration by parts:

(a) Determine indefinite integrals. **BC ONLY**

(b) Evaluate definite integrals. **BC ONLY**

ESSENTIAL KNOWLEDGE

FUN-6.E.1

Integration by parts is a technique for finding antiderivatives. **BC ONLY**

SUGGESTED SKILL

Implementing Mathematical Processes

1.E

Apply appropriate mathematical rules or procedures, with and without technology.

TOPIC 6.12

Integrating Using Linear Partial Fractions BC ONLY

Required Course Content

ENDURING UNDERSTANDING

FUN-6

Recognizing opportunities to apply knowledge of geometry and mathematical rules can simplify integration.

LEARNING OBJECTIVE

FUN-6.F

For integrands requiring integration by linear partial fractions:

(a) Determine indefinite integrals. **BC ONLY**

(b) Evaluate definite integrals. **BC ONLY**

ESSENTIAL KNOWLEDGE

FUN-6.F.1

Some rational functions can be decomposed into sums of ratios of linear, nonrepeating factors to which basic integration techniques can be applied. **BC ONLY**

TOPIC 6.13

Evaluating Improper Integrals BC ONLY

SUGGESTED SKILL

Implementing Mathematical Processes

1.E

Apply appropriate mathematical rules or procedures, with and without technology.

AVAILABLE RESOURCE

- AP Online Teacher Community Discussion > **U-Substitution with Improper Integrals**

Required Course Content

ENDURING UNDERSTANDING

LIM-6

The use of limits allows us to show that the areas of unbounded regions may be finite.

LEARNING OBJECTIVE

LIM-6.A

Evaluate an improper integral or determine that the integral diverges. **BC ONLY**

ESSENTIAL KNOWLEDGE

LIM-6.A.1

An improper integral is an integral that has one or both limits infinite or has an integrand that is unbounded in the interval of integration. **BC ONLY**

LIM-6.A.2

Improper integrals can be determined using limits of definite integrals. **BC ONLY**

SUGGESTED SKILL

Implementing Mathematical Processes

1.C

Identify an appropriate mathematical rule or procedure based on the classification of a given expression.

TOPIC 6.14

Selecting Techniques for Antidifferentiation

This topic is intended to focus on the skill of selecting an appropriate procedure for antidifferentiation. Students should be given opportunities to practice when and how to apply all learning objectives relating to antidifferentiation.

AP CALCULUS AB AND BC

UNIT 7

Differential Equations

AP EXAM WEIGHTING	6–12% AB	6–9% BC
CLASS PERIODS	~8–9 AB	~9–10 BC

Remember to go to **AP Classroom** to assign students the online **Personal Progress Check** for this unit.

Whether assigned as homework or completed in class, the **Personal Progress Check** provides each student with immediate feedback related to this unit's topics and skills.

Personal Progress Check 7

Multiple-choice:
~15 questions (AB)
~20 questions (BC)
Free-response: 3 questions

UNIT 7

AP EXAM WEIGHTING	6–12% AB	6–9% BC
CLASS PERIODS	~8–9 AB	~9–10 BC

Differential Equations

BIG IDEA 3
Analysis of Functions **FUN**

- How can we derive a model for the number of computers, C, infected by a virus, given a model for how fast the computers are being infected, $\frac{dC}{dt}$, at a particular time?

Developing Understanding

In this unit, students will learn to set up and solve separable differential equations. Slope fields can be used to represent solution curves to a differential equation and build understanding that there are infinitely many general solutions to a differential equation, varying only by a constant of integration. Students can locate a unique solution relevant to a particular situation, provided they can locate a point on the solution curve. By writing and solving differential equations leading to models for exponential growth and decay and logistic growth **BC ONLY**, students build understanding of topics introduced in Algebra II and other courses.

Building the Mathematical Practices

1.E 2.C 3.G 4.D

In this unit, students will translate mathematical information from one representation to another by matching equations and slope fields, rewriting verbal statements as differential equations, and sketching slope fields that match their symbolic representations. Provide students with explicit guidance on how to select an appropriate graphing technique. As students practice Euler's method, encourage them to transfer skills using tangent line approximations, rather than simply memorizing an algorithm **BC ONLY**.

Because the problems in this unit model real-world scenarios, help students to develop proficiency in transferring the mathematical procedures they've learned in "x's and y's" to equivalent environments with variable names other than x, y, and t. Using differentiation to confirm that solutions to differential equations are accurate and appropriate also helps students to develop an understanding of what it means to say that an equation is a solution to a differential equation.

Preparing for the AP Exam

Students should practice setting up and solving contextual questions involving separable differential equations until the solution strategy becomes routine: separate variables, antidifferentiate both sides of the equation and add a constant of integration, use initial conditions to determine the constant of integration, and rearrange the resulting expression to complete the solution. Failure to separate variables or omitting the constant of integration severely limits the number of points a student can earn on the AP Exam. A common error in antidifferentiation is to assume that all differential equations involving fractions have logarithmic solutions, presumably because some do.

Students should learn to recognize the forms of differential equations resulting in exponential and logistic **BC ONLY** models. These may be used or interpreted without performing the derivation. Students should also be reminded that differential equations give us information about the derivative and may be used directly to find information about a slope or rate of change.

UNIT AT A GLANCE

Enduring Understanding	Topic	Suggested Skills	Class Periods ~8–9 CLASS PERIODS (AB) ~9–10 CLASS PERIODS (BC)
FUN-7	**7.1 Modeling Situations with Differential Equations**	2.C Identify a re-expression of mathematical information presented in a given representation.	
	7.2 Verifying Solutions for Differential Equations	3.G Confirm that solutions are accurate and appropriate.	
	7.3 Sketching Slope Fields	2.C Identify a re-expression of mathematical information presented in a given representation.	
	7.4 Reasoning Using Slope Fields	4.D Use appropriate graphing techniques.	
	7.5 Approximating Solutions Using Euler's Method BC ONLY	1.E Apply appropriate mathematical rules or procedures, with and without technology.	
	7.6 Finding General Solutions Using Separation of Variables	1.E Apply appropriate mathematical rules or procedures, with and without technology.	
	7.7 Finding Particular Solutions Using Initial Conditions and Separation of Variables	1.E Apply appropriate mathematical rules or procedures, with and without technology.	
	7.8 Exponential Models with Differential Equations	3.G Confirm that solutions are accurate and appropriate.	
	7.9 Logistic Models with Differential Equations BC ONLY	3.F Explain the meaning of mathematical solutions in context.	

AP

Go to **AP Classroom** to assign the **Personal Progress Check** for Unit 7. Review the results in class to identify and address any student misunderstandings.

SAMPLE INSTRUCTIONAL ACTIVITIES

The sample activities on this page are optional and are offered to provide possible ways to incorporate various instructional approaches into the classroom. Teachers do not need to use these activities or instructional approaches and are free to alter or edit them. The examples below were developed in partnership with teachers from the AP community to share ways that they approach teaching some of the topics in this unit. Please refer to the Instructional Approaches section beginning on p. 199 for more examples of activities and strategies.

Activity	Topic	Sample Activity
1	7.3	**Match Mine** Give student pairs a blank 3 × 3 game board and nine graphs of slope fields, each on a separate card. Some should be in terms of x only, some in terms of y only, and some in terms of x and y. Be sure to include at least one trigonometric function. Student A arranges the graphs on the grid without showing Student B and then describes the arrangement so Student B can try to match it on their own board.
2	7.6	**Numbered Heads Together** Have each student complete the same problem individually (e.g., $y' = 2xy^2$, $\frac{dy}{dx} = y^2 + 1$, or $3y\,dy = (x^2+1)dx$). Make sure to use a variety of notation in whatever problem you pick. Then have students compare answers and procedures within groups. Students fix any mistakes until they all agree on the same answer.
3	7.7 7.8	**Collaborative Poster** Assign each student a role within their group: • Separating the variables • Integrating both sides • Finding C • Writing the final particular solution Then distribute a free-response question to each group and have them work on their assigned roles to solve the problem together. Examples include the following: • **2002 Form B #5(b)** (not transcendental) • **2011 #5(c)** (transcendental) • **2012 #5(c)** (transcendental) • **2014 #6(c)** (transcendental)

SUGGESTED SKILL

Connecting Representations

2.C

Identify a re-expression of mathematical information presented in a given representation.

AVAILABLE RESOURCE

- Classroom Resource > **Differential Equations**

TOPIC 7.1

Modeling Situations with Differential Equations

Required Course Content

ENDURING UNDERSTANDING

FUN-7

Solving differential equations allows us to determine functions and develop models.

LEARNING OBJECTIVE

FUN-7.A

Interpret verbal statements of problems as differential equations involving a derivative expression.

ESSENTIAL KNOWLEDGE

FUN-7.A.1

Differential equations relate a function of an independent variable and the function's derivatives.

TOPIC 7.2

Verifying Solutions for Differential Equations

SUGGESTED SKILL

Justification

3.G

Confirm that solutions are accurate and appropriate.

AVAILABLE RESOURCE

- Classroom Resource > **Differential Equations**

Required Course Content

ENDURING UNDERSTANDING

FUN-7

Solving differential equations allows us to determine functions and develop models.

LEARNING OBJECTIVE

FUN-7.B

Verify solutions to differential equations.

ESSENTIAL KNOWLEDGE

FUN-7.B.1

Derivatives can be used to verify that a function is a solution to a given differential equation.

FUN-7.B.2

There may be infinitely many general solutions to a differential equation.

SUGGESTED SKILL

Connecting Representations

2.C

Identify a re-expression of mathematical information presented in a given representation.

AVAILABLE RESOURCES

- Classroom Resource > **Slope Fields**
- Classroom Resource > **Differential Equations**

TOPIC 7.3
Sketching Slope Fields

Required Course Content

ENDURING UNDERSTANDING

FUN-7

Solving differential equations allows us to determine functions and develop models.

LEARNING OBJECTIVE

FUN-7.C

Estimate solutions to differential equations.

ESSENTIAL KNOWLEDGE

FUN-7.C.1

A slope field is a graphical representation of a differential equation on a finite set of points in the plane.

FUN-7.C.2

Slope fields provide information about the behavior of solutions to first-order differential equations.

TOPIC 7.4
Reasoning Using Slope Fields

SUGGESTED SKILL

Communication and Notation

4.D

Use appropriate graphing techniques.

AVAILABLE RESOURCES

- Classroom Resource > **Slope Fields**
- Classroom Resource > **Differential Equations**

Required Course Content

ENDURING UNDERSTANDING

FUN-7

Solving differential equations allows us to determine functions and develop models.

LEARNING OBJECTIVE

FUN-7.C

Estimate solutions to differential equations.

ESSENTIAL KNOWLEDGE

FUN-7.C.3

Solutions to differential equations are functions or families of functions.

SUGGESTED SKILL

Implementing Mathematical Processes

1.E

Apply appropriate mathematical rules or procedures, with and without technology.

AVAILABLE RESOURCES

- Classroom Resource > **Approximation**
- Classroom Resource > **Differential Equations**

TOPIC 7.5

Approximating Solutions Using Euler's Method BC ONLY

Required Course Content

ENDURING UNDERSTANDING

FUN-7

Solving differential equations allows us to determine functions and develop models.

LEARNING OBJECTIVE

FUN-7.C

Estimate solutions to differential equations.

ESSENTIAL KNOWLEDGE

FUN-7.C.4

Euler's method provides a procedure for approximating a solution to a differential equation or a point on a solution curve. **BC ONLY**

TOPIC 7.6
Finding General Solutions Using Separation of Variables

Required Course Content

ENDURING UNDERSTANDING

FUN-7

Solving differential equations allows us to determine functions and develop models.

LEARNING OBJECTIVE

FUN-7.D

Determine general solutions to differential equations.

ESSENTIAL KNOWLEDGE

FUN-7.D.1

Some differential equations can be solved by separation of variables.

FUN-7.D.2

Antidifferentiation can be used to find general solutions to differential equations.

SUGGESTED SKILL

Implementing Mathematical Processes

1.E

Apply appropriate mathematical rules or procedures, with and without technology.

AVAILABLE RESOURCE

- Classroom Resource > **Differential Equations**

SUGGESTED SKILL

Implementing Mathematical Processes

1.E

Apply appropriate mathematical rules or procedures, with and without technology.

AVAILABLE RESOURCE

- Classroom Resource > **Differential Equations**

TOPIC 7.7
Finding Particular Solutions Using Initial Conditions and Separation of Variables

Required Course Content

ENDURING UNDERSTANDING

FUN-7

Solving differential equations allows us to determine functions and develop models.

LEARNING OBJECTIVE

FUN-7.E

Determine particular solutions to differential equations.

ESSENTIAL KNOWLEDGE

FUN-7.E.1

A general solution may describe infinitely many solutions to a differential equation. There is only one particular solution passing through a given point.

FUN-7.E.2

The function F defined by $F(x) = y_0 + \int_a^x f(t)\,dt$ is a particular solution to the differential equation $\frac{dy}{dx} = f(x)$, satisfying $F(a) = y_0$.

FUN-7.E.3

Solutions to differential equations may be subject to domain restrictions.

TOPIC 7.8

Exponential Models with Differential Equations

SUGGESTED SKILL

Justification

3.G

Confirm that solutions are accurate and appropriate.

Required Course Content

ENDURING UNDERSTANDING

FUN-7

Solving differential equations allows us to determine functions and develop models.

LEARNING OBJECTIVE

FUN-7.F

Interpret the meaning of a differential equation and its variables in context.

ESSENTIAL KNOWLEDGE

FUN-7.F.1

Specific applications of finding general and particular solutions to differential equations include motion along a line and exponential growth and decay.

FUN-7.F.2

The model for exponential growth and decay that arises from the statement "The rate of change of a quantity is proportional to the size of the quantity" is $\frac{dy}{dt} = ky$.

LEARNING OBJECTIVE

FUN-7.G

Determine general and particular solutions for problems involving differential equations in context.

ESSENTIAL KNOWLEDGE

FUN-7.G.1

The exponential growth and decay model, $\frac{dy}{dt} = ky$, with initial condition $y = y_0$ when $t = 0$, has solutions of the form $y = y_0 e^{kt}$.

SUGGESTED SKILL

Justification

3.F

Explain the meaning of mathematical solutions in context.

TOPIC 7.9

Logistic Models with Differential Equations BC ONLY

Required Course Content

ENDURING UNDERSTANDING

FUN-7

Solving differential equations allows us to determine functions and develop models.

LEARNING OBJECTIVE

FUN-7.H

Interpret the meaning of the logistic growth model in context. **BC ONLY**

ESSENTIAL KNOWLEDGE

FUN-7.H.1

The model for logistic growth that arises from the statement "The rate of change of a quantity is jointly proportional to the size of the quantity and the difference between the quantity and the carrying capacity" is $\frac{dy}{dt} = ky(a - y)$. **BC ONLY**

FUN-7.H.2

The logistic differential equation and initial conditions can be interpreted without solving the differential equation. **BC ONLY**

FUN-7.H.3

The limiting value (carrying capacity) of a logistic differential equation as the independent variable approaches infinity can be determined using the logistic growth model and initial conditions. **BC ONLY**

FUN-7.H.4

The value of the dependent variable in a logistic differential equation at the point when it is changing fastest can be determined using the logistic growth model and initial conditions. **BC ONLY**

AP CALCULUS AB AND BC

UNIT 8

Applications of Integration

AP EXAM WEIGHTING	10–15% AB	6–9% BC
CLASS PERIODS	~19–20 AB	~13–14 BC

Remember to go to **AP Classroom** to assign students the online **Personal Progress Check** for this unit.

Whether assigned as homework or completed in class, the **Personal Progress Check** provides each student with immediate feedback related to this unit's topics and skills.

Personal Progress Check 8

Multiple-choice: ~30 questions
Free-response: 3 questions

UNIT 8

	AB	BC
AP EXAM WEIGHTING	10–15% AB	6–9% BC
CLASS PERIODS	~19–20 AB	~13–14 BC

Applications of Integration

BIG IDEA 1
Change **CHA**

- How is finding the number of visitors to a museum over an interval of time based on information about the rate of entry similar to finding the area of a region between a curve and the x-axis?

Developing Understanding

In this unit, students will learn how to find the average value of a function, model particle motion and net change, and determine areas, volumes, and lengths **BC ONLY** defined by the graphs of functions. Emphasis should be placed on developing an understanding of integration that can be transferred across these and many other applications. Understanding that the area, volume, and length **BC ONLY** problems studied in this unit are limiting cases of Riemann sums of rectangle areas, prism volumes, or segment lengths **BC ONLY** saves students from memorizing a long list of seemingly unrelated formulas and develops meaningful understanding of integration.

Building the Mathematical Practices

1.D 2.D 3.D 4.C

As in Unit 4, students will need to practice interpreting verbal scenarios, extracting relevant mathematical information, selecting an appropriate procedure, and then applying that procedure correctly and interpreting their solution in the context of the problem. Now that students have been exposed to application problems involving both differentiation and antidifferentiation, some may struggle to determine which procedure is applicable. Walk students through different types of scenarios and explain the underlying reasons why some situations call for differentiation while others call for integration.

This unit also involves geometric applications of integration. When using the disc and washer methods, focusing on orientation (i.e., horizontal or vertical) will help students determine whether the "thickness" is with respect to x or y. Students should practice solving variations on these calculus-based geometry problems until they can decide which variable to integrate with respect to without prompting. Relating graphical representations to symbolic representations, such as Riemann sums and definite integrals, develops these skills and helps students to master the content.

Preparing for the AP Exam

On the AP Exam, students need to identify relevant information conveyed in various representations. Key words, such as "accumulation" or "net change," help to identify mathematical structures and corresponding solution strategies. Some students confuse the average value and the average rate of change of a function on an interval. To alleviate confusion, first provide students with average value problems accompanied by relevant graphs and guide them to an understanding of why an average value may be less than, equal to, or greater than the midpoint of the range. Then review average rate of change problems from Unit 2 and present students with free-response questions that will allow them to practice distinguishing between average value and average rate of change problems.

In free-response questions, continue to require students to show supporting work by presenting a correct expression using appropriate notation and the mathematical structures of solutions, as in $V = \pi\int_1^4 \left[\left(f(x)-3\right)^2 - \left(g(x)-3\right)^2\right]dx$, for example.

UNIT AT A GLANCE

Enduring Understanding	Topic	Suggested Skills	Class Periods ~19–20 CLASS PERIODS (AB) ~13–14 CLASS PERIODS (BC)
CHA-4	**8.1 Finding the Average Value of a Function on an Interval**	1.E Apply appropriate mathematical rules or procedures, with and without technology.	
	8.2 Connecting Position, Velocity, and Acceleration of Functions Using Integrals	1.D Identify an appropriate mathematical rule or procedure based on the relationship between concepts (e.g., rate of change and accumulation) or processes (e.g., differentiation and its inverse process, anti-differentiation) to solve problems.	
	8.3 Using Accumulation Functions and Definite Integrals in Applied Contexts	3.D Apply an appropriate mathematical definition, theorem, or test.	
CHA-5	**8.4 Finding the Area Between Curves Expressed as Functions of x**	4.C Use appropriate mathematical symbols and notation (e.g., Represent a derivative using $f'(x)$, y', and $\frac{dy}{dx}$).	
	8.5 Finding the Area Between Curves Expressed as Functions of y	1.E Apply appropriate mathematical rules or procedures, with and without technology.	
	8.6 Finding the Area Between Curves That Intersect at More Than Two Points	2.B Identify mathematical information from graphical, numerical, analytical, and/or verbal representations.	
	8.7 Volumes with Cross Sections: Squares and Rectangles	3.D Apply an appropriate mathematical definition, theorem, or test.	
	8.8 Volumes with Cross Sections: Triangles and Semicircles	3.D Apply an appropriate mathematical definition, theorem, or test.	
	8.9 Volume with Disc Method: Revolving Around the x- or y-Axis	3.D Apply an appropriate mathematical definition, theorem, or test.	

continued on next page

UNIT AT A GLANCE *(cont'd)*

Enduring Understanding	Topic	Suggested Skills	Class Periods ~19–20 CLASS PERIODS (AB) ~13–14 CLASS PERIODS (BC)
CHA-5	**8.10 Volume with Disc Method: Revolving Around Other Axes**	2.D Identify how mathematical characteristics or properties of functions are related in different representations.	
	8.11 Volume with Washer Method: Revolving Around the x- or y-Axis	4.E Apply appropriate rounding procedures.	
	8.12 Volume with Washer Method: Revolving Around Other Axes	2.D Identify how mathematical characteristics or properties of functions are related in different representations.	
CHA-6	**8.13 The Arc Length of a Smooth, Planar Curve and Distance Traveled** BC ONLY	3.D Apply an appropriate mathematical definition, theorem, or test.	

Go to **AP Classroom** to assign the **Personal Progress Check** for Unit 8. Review the results in class to identify and address any student misunderstandings.

SAMPLE INSTRUCTIONAL ACTIVITIES

The sample activities on this page are optional and are offered to provide possible ways to incorporate various instructional approaches into the classroom. Teachers do not need to use these activities or instructional approaches and are free to alter or edit them. The examples below were developed in partnership with teachers from the AP community to share ways that they approach teaching some of the topics in this unit. Please refer to the Instructional Approaches section beginning on p. 199 for more examples of activities and strategies.

Activity	Topic	Sample Activity
1	8.1	**Scavenger Hunt** Post around the room 8–10 problem cards, each of which also includes a solution to a previous problem. Include average value problems with tables and functions; also include tables that have values from 0 to 16, for example, but where the average value is only on the interval from 0 to 12. Card 1 could say: Find the average value of $f(x) = \sin(x)$ on the interval $[0, \pi]$. A second card would have the answer to that card $\left(\frac{2}{\pi}\right)$ along with a new question: Find the average value of $f(x) = 3x^2 - 3$ on $[1, 3]$. A third card would have that answer (10) and so on. The answer to the last card goes on top of the first card.
2	8.6	**Round Table** In groups of four, each student has an identical paper with the same free-response question (e.g., 2015 AB #2(a)), along with four labeled boxes representing steps in the problem: ▪ Identify all points of intersection. ▪ Set up the integral(s). ▪ Integrate by hand. ▪ Integrate using a calculator. Have students complete the first step on their paper, and then pass the paper clockwise to another member in their group. That student checks the first step and then completes the second step on the paper. Students rotate again and the process continues until each student has their own paper back.
3	8.9 8.10 8.11 8.12	**Quiz-Quiz-Trade** Create cards with problems revolving around the x- or y-axis and others revolving around other axes (e.g., $y = x$ or $y = 3$). Give each student a card and have them write their answer on the back. Students quiz a partner about their own card then switch cards and repeat the process with a new partner. For the first round, concentrate on just setting up the integrals (e.g., 2009 AB Form B #4(c), 2010 AB/BC #4(b), 2011 AB #3(c), and 2013 AB #5(b)). In the second round, students can use their calculators to find the volume (e.g., 2001 AB #1(c), 2006 AB/BC #1(b), 2007 AB/BC #1(b), and 2008 AB Form B #1(b)).

TOPIC 8.1

Finding the Average Value of a Function on an Interval

Required Course Content

ENDURING UNDERSTANDING

CHA-4

Definite integrals allow us to solve problems involving the accumulation of change over an interval.

LEARNING OBJECTIVE

CHA-4.B

Determine the average value of a function using definite integrals.

ESSENTIAL KNOWLEDGE

CHA-4.B.1

The average value of a continuous function f over an interval $[a, b]$ is $\frac{1}{b-a}\int_a^b f(x)dx$.

SUGGESTED SKILL

Implementing Mathematical Processes

1.E

Apply appropriate mathematical rules or procedures, with and without technology.

AVAILABLE RESOURCE

- Professional Development > **Interpreting Context for Definite Integrals**

SUGGESTED SKILL

Implementing Mathematical Processes

1.D

Identify an appropriate mathematical rule or procedure based on the relationship between concepts or processes to solve problems.

AVAILABLE RESOURCE

- Classroom Resource > **Motion**

TOPIC 8.2

Connecting Position, Velocity, and Acceleration of Functions Using Integrals

Required Course Content

ENDURING UNDERSTANDING

CHA-4

Definite integrals allow us to solve problems involving the accumulation of change over an interval.

LEARNING OBJECTIVE

CHA-4.C

Determine values for positions and rates of change using definite integrals in problems involving rectilinear motion.

ESSENTIAL KNOWLEDGE

CHA-4.C.1

For a particle in rectilinear motion over an interval of time, the definite integral of velocity represents the particle's displacement over the interval of time, and the definite integral of speed represents the particle's total distance traveled over the interval of time.

TOPIC 8.3

Using Accumulation Functions and Definite Integrals in Applied Contexts

Required Course Content

ENDURING UNDERSTANDING

CHA-4

Definite integrals allow us to solve problems involving the accumulation of change over an interval.

LEARNING OBJECTIVE	ESSENTIAL KNOWLEDGE
CHA-4.D Interpret the meaning of a definite integral in accumulation problems.	**CHA-4.D.1** A function defined as an integral represents an accumulation of a rate of change. **CHA-4.D.2** The definite integral of the rate of change of a quantity over an interval gives the net change of that quantity over that interval.
CHA-4.E Determine net change using definite integrals in applied contexts.	**CHA-4.E.1** The definite integral can be used to express information about accumulation and net change in many applied contexts.

SUGGESTED SKILL

Justification

3.D

Apply an appropriate mathematical definition, theorem, or test.

AVAILABLE RESOURCE

- Professional Development > **Interpreting Context for Definite Integrals**

SUGGESTED SKILL

Communication and Notation

4.C

Use appropriate mathematical symbols and notation.

TOPIC 8.4

Finding the Area Between Curves Expressed as Functions of x

Required Course Content

ENDURING UNDERSTANDING

CHA-5

Definite integrals allow us to solve problems involving the accumulation of change in area or volume over an interval.

LEARNING OBJECTIVE

CHA-5.A

Calculate areas in the plane using the definite integral.

ESSENTIAL KNOWLEDGE

CHA-5.A.1

Areas of regions in the plane can be calculated with definite integrals.

TOPIC 8.5

Finding the Area Between Curves Expressed as Functions of y

SUGGESTED SKILL

Implementing Mathematical Processes

1.E

Apply appropriate mathematical rules or procedures, with and without technology.

Required Course Content

ENDURING UNDERSTANDING

CHA-5

Definite integrals allow us to solve problems involving the accumulation of change in area or volume over an interval.

LEARNING OBJECTIVE

CHA-5.A

Calculate areas in the plane using the definite integral.

ESSENTIAL KNOWLEDGE

CHA-5.A.2

Areas of regions in the plane can be calculated using functions of either x or y.

SUGGESTED SKILL

Connecting Representations

2.B

Identify mathematical information from graphical, numerical, analytical, and/or verbal representations.

TOPIC 8.6

Finding the Area Between Curves That Intersect at More Than Two Points

Required Course Content

ENDURING UNDERSTANDING

CHA-5

Definite integrals allow us to solve problems involving the accumulation of change in area or volume over an interval.

LEARNING OBJECTIVE

CHA-5.A

Calculate areas in the plane using the definite integral.

ESSENTIAL KNOWLEDGE

CHA-5.A.3

Areas of certain regions in the plane may be calculated using a sum of two or more definite integrals or by evaluating a definite integral of the absolute value of the difference of two functions.

TOPIC 8.7

Volumes with Cross Sections: Squares and Rectangles

SUGGESTED SKILL

Justification

3.D

Apply an appropriate mathematical definition, theorem, or test.

Required Course Content

ENDURING UNDERSTANDING

CHA-5

Definite integrals allow us to solve problems involving the accumulation of change in area or volume over an interval.

LEARNING OBJECTIVE

CHA-5.B

Calculate volumes of solids with known cross sections using definite integrals.

ESSENTIAL KNOWLEDGE

CHA-5.B.1

Volumes of solids with square and rectangular cross sections can be found using definite integrals and the area formulas for these shapes.

SUGGESTED SKILL

Justification

3.D

Apply an appropriate mathematical definition, theorem, or test.

ILLUSTRATIVE EXAMPLES

- Illustrative examples of other cross sections in CHA-5.B.3:
- * The volume of a funnel whose cross sections are circles can be found using the area formula for a circle and definite integrals (see **2016 AB Exam FRQ #5(b)**).
- * The volume of a solid whose cross sectional area is defined using a function can be found using the known area function and a definite integral (see **2009 AB Exam FRQ #4(c)**).

TOPIC 8.8

Volumes with Cross Sections: Triangles and Semicircles

Required Course Content

ENDURING UNDERSTANDING

CHA-5

Definite integrals allow us to solve problems involving the accumulation of change in area or volume over an interval.

LEARNING OBJECTIVE

CHA-5.B

Calculate volumes of solids with known cross sections using definite integrals.

ESSENTIAL KNOWLEDGE

CHA-5.B.2

Volumes of solids with triangular cross sections can be found using definite integrals and the area formulas for these shapes.

CHA-5.B.3

Volumes of solids with semicircular and other geometrically defined cross sections can be found using definite integrals and the area formulas for these shapes.

TOPIC 8.9

Volume with Disc Method: Revolving Around the x- or y-Axis

Required Course Content

ENDURING UNDERSTANDING

CHA-5

Definite integrals allow us to solve problems involving the accumulation of change in area or volume over an interval.

LEARNING OBJECTIVE

CHA-5.C

Calculate volumes of solids of revolution using definite integrals.

ESSENTIAL KNOWLEDGE

CHA-5.C.1

Volumes of solids of revolution around the x- or y-axis may be found by using definite integrals with the disc method.

SUGGESTED SKILL

Justification

3.D

Apply an appropriate mathematical definition, theorem, or test.

SUGGESTED SKILL

Connecting Representations

2.D

Identify how mathematical characteristics or properties of functions are related in different representations.

TOPIC 8.10

Volume with Disc Method: Revolving Around Other Axes

Required Course Content

ENDURING UNDERSTANDING

CHA-5

Definite integrals allow us to solve problems involving the accumulation of change in area or volume over an interval.

LEARNING OBJECTIVE

CHA-5.C

Calculate volumes of solids of revolution using definite integrals.

ESSENTIAL KNOWLEDGE

CHA-5.C.2

Volumes of solids of revolution around any horizontal or vertical line in the plane may be found by using definite integrals with the disc method.

TOPIC 8.11

Volume with Washer Method: Revolving Around the x- or y-Axis

Required Course Content

ENDURING UNDERSTANDING

CHA-5

Definite integrals allow us to solve problems involving the accumulation of change in area or volume over an interval.

LEARNING OBJECTIVE

CHA-5.C

Calculate volumes of solids of revolution using definite integrals.

ESSENTIAL KNOWLEDGE

CHA-5.C.3

Volumes of solids of revolution around the x- or y-axis whose cross sections are ring shaped may be found using definite integrals with the washer method.

SUGGESTED SKILL

Communication and Notation

4.E

Apply appropriate rounding procedures.

SUGGESTED SKILL

Connecting Representations

2.D

Identify how mathematical characteristics or properties of functions are related in different representations.

AVAILABLE RESOURCE

- Classroom Resource > **Volumes of Solids of Revolution**

TOPIC 8.12

Volume with Washer Method: Revolving Around Other Axes

Required Course Content

ENDURING UNDERSTANDING

CHA-5

Definite integrals allow us to solve problems involving the accumulation of change in area or volume over an interval.

LEARNING OBJECTIVE

CHA-5.C

Calculate volumes of solids of revolution using definite integrals.

ESSENTIAL KNOWLEDGE

CHA-5.C.4

Volumes of solids of revolution around any horizontal or vertical line whose cross sections are ring shaped may be found using definite integrals with the washer method.

TOPIC 8.13

The Arc Length of a Smooth, Planar Curve and Distance Traveled BC ONLY

SUGGESTED SKILL

Justification

3.D

Apply an appropriate mathematical definition, theorem, or test.

Required Course Content

ENDURING UNDERSTANDING

CHA-6

Definite integrals allow us to solve problems involving the accumulation of change in length over an interval.

LEARNING OBJECTIVE

CHA-6.A

Determine the length of a curve in the plane defined by a function, using a definite integral. **BC ONLY**

ESSENTIAL KNOWLEDGE

CHA-6.A.1

The length of a planar curve defined by a function can be calculated using a definite integral. **BC ONLY**

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AP CALCULUS AB AND BC

UNIT 9 BC ONLY

Parametric Equations, Polar Coordinates, and Vector-Valued Functions

AP EXAM WEIGHTING **11–12%** BC ONLY

CLASS PERIODS **~10–11**

Remember to go to **AP Classroom** to assign students the online **Personal Progress Check** for this unit.

Whether assigned as homework or completed in class, the **Personal Progress Check** provides each student with immediate feedback related to this unit's topics and skills.

Personal Progress Check 9

Multiple-choice: ~25 questions
Free-response: 3 questions

UNIT 9

AP EXAM WEIGHTING	~11–12% BC
CLASS PERIODS	~10–11 BC

Parametric Equations, Polar Coordinates, and Vector-Valued Functions

BIG IDEA 1
***Change* CHA**
- How can we model motion not constrained to a linear path?

BIG IDEA 3
***Analysis of Functions* FUN**
- How does the chain rule help us to analyze graphs defined using parametric equations or polar functions?

Developing Understanding

In this unit, students will build on their understanding of straight-line motion to solve problems in which particles are moving along curves in the plane. Students will define parametric equations and vector-valued functions to describe planar motion and apply calculus to solve motion problems. Students will learn that polar equations are a special case of parametric equations and will apply calculus to analyze graphs and determine lengths and areas. This unit should be treated as an opportunity to reinforce past learning and transfer knowledge and skills to new situations, rather than as a new list of facts or strategies to memorize.

Building the Mathematical Practices

1.D 1.E 2.D 3.D

As students transition to parametric and vector-valued functions, they'll need to practice previously learned concepts and skills to reinforce the new procedures and representations they're learning in Unit 9. As with particle motion on a line, students learning to handle motion in the plane will need to practice interpreting which procedure is needed for different scenarios (differentiation or integration) and solving for speed, velocity, distance traveled, or initial position.

Reinforce the importance of precise notation, particularly regarding the variable of differentiation, as well as correct application of the chain rule. Leibniz notation helps students to remember how to find the derivative of y with respect to x for coordinates defined using the parameter t:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{dy}{dt} \cdot \frac{dt}{dx}, \text{ provided } \frac{dx}{dt} \neq 0.$$

Since $\frac{dy}{dx}$ is in terms of t, students must be particularly careful when determining $\frac{d^2y}{dx^2}$.

Similarly, using definite integrals to represent lengths and areas defined by polar curves is based on the same principles as calculating lengths and areas defined by the graphs of more familiar functions (i.e., the limit of a Riemann sum). Students will need to practice with trigonometric identities, radian measures and formulas for arc length and area of a sector to reinforce practice with associated calculus topics.

Preparing for the AP Exam

While students need more experience shifting mindsets from rectangular to polar coordinate systems, errors in arithmetic, algebra, trigonometry, and procedures such as the chain rule are often even more problematic. Provide opportunities for students to reinforce familiar skills and concepts as they practice new techniques in preparation for the AP Exam. As with analysis of graphs, sign charts can be useful tools for identifying answers to questions about the direction of motion or whether speed is increasing or decreasing, for example. To earn points for justification, however, students must connect their work to a relevant definition or theorem, as in the **Scoring Guidelines for 2017 AB5**. Continue to emphasize accounting for initial values, as in past units, as well as precise communication and notational fluency. Paying attention to subscripts in problems involving more than one particle is essential to clear communication.

UNIT AT A GLANCE

Enduring Understanding	Topic	Suggested Skills	Class Periods ~10–11 CLASS PERIODS
CHA-3	**9.1 Defining and Differentiating Parametric Equations**	2.D Identify how mathematical characteristics or properties of functions are related in different representations.	
	9.2 Second Derivatives of Parametric Equations	1.E Apply appropriate mathematical rules or procedures, with and without technology.	
CHA-6	**9.3 Finding Arc Lengths of Curves Given by Parametric Equations**	1.D Identify an appropriate mathematical rule or procedure based on the relationship between concepts (e.g., rate of change and accumulation) or processes (e.g., differentiation and its inverse process, anti-differentiation) to solve problems.	
CHA-3	**9.4 Defining and Differentiating Vector-Valued Functions**	1.D Identify an appropriate mathematical rule or procedure based on the relationship between concepts (e.g., rate of change and accumulation) or processes (e.g., differentiation and its inverse process, anti-differentiation) to solve problems.	
FUN-8	**9.5 Integrating Vector-Valued Functions**	1.E Apply appropriate mathematical rules or procedures, with and without technology.	
	9.6 Solving Motion Problems Using Parametric and Vector-Valued Functions	1.E Apply appropriate mathematical rules or procedures, with and without technology.	
FUN-3	**9.7 Defining Polar Coordinates and Differentiating in Polar Form**	2.D Identify how mathematical characteristics or properties of functions are related in different representations.	
CHA-5	**9.8 Find the Area of a Polar Region or the Area Bounded by a Single Polar Curve**	3.D Apply an appropriate mathematical definition, theorem, or test.	
	9.9 Finding the Area of the Region Bounded by Two Polar Curves	3.D Apply an appropriate mathematical definition, theorem, or test.	

Go to **AP Classroom** to assign the **Personal Progress Check** for Unit 9. Review the results in class to identify and address any student misunderstandings.

SAMPLE INSTRUCTIONAL ACTIVITIES

The sample activities on this page are optional and are offered to provide possible ways to incorporate various instructional approaches into the classroom. Teachers do not need to use these activities or instructional approaches and are free to alter or edit them. The examples below were developed in partnership with teachers from the AP community to share ways that they approach teaching some of the topics in this unit. Please refer to the Instructional Approaches section beginning on p. 199 for more examples of activities and strategies.

Activity	Topic	Sample Activity
1	9.2	**Numbered Heads Together** Organize the class into groups of four and give each student a number. Give the class one set of parametric equations and have students individually find the second derivative. When all students in a group have finished, have them stand up to discuss their answers, make any necessary corrections, and then sit back down. Choose a group and call a specific student number so that the student can share the answer with the class.
2	9.5 9.6	**Scavenger Hunt** Create cards containing a parametric initial value question and the answer to a different question. Students can start at any question, moving around the room to make a full circuit of all questions once complete. Focus on giving students an initial value and asking for one or both component values at a different t- or x-value (e.g., $x'(t) = 2t$, $y'(t) = \cos(t)$, and position is $(2, 0)$ at $t = \pi$. Find position at $t = \frac{5\pi}{6}$.).
3	9.8	**Create Representations** Give students polar equations of various types: circle, limaçon with inner loop, cardioid, dimpled limaçon, rose curve, or lemniscates. Have students create a table of values, sketch the graph of the curve using rectangular coordinates, and sketch the graph of the curve using polar coordinates. Preface this activity by modeling the steps with one function on large paper, using wiki sticks to show the y-values as heights becoming the r-values as radii.
4	9.8	**Paraphrasing** Give students a proof which derives the polar area formula. Ask them to restate the meaning and derivation of this formula in their own words. Have them also compare and contrast this formula to the integration used to find areas under functions in the Cartesian coordinate system.
5	9.9	**Stand Up, Hand Up, Pair Up** Give each pair of students a polar area question to solve. Once they have completed both roles obtaining only the integral setup, have them use a calculator to find the numeric solution and confirm with you before standing up to switch pairs.

SUGGESTED SKILL

Connecting Representations

2.D

Identify how mathematical characteristics or properties of functions are related in different representations.

AVAILABLE RESOURCES

- Classroom Resource > **Vectors**
- External Resource > **Davidson Next**

TOPIC 9.1
Defining and Differentiating Parametric Equations

Required Course Content

ENDURING UNDERSTANDING

CHA-3

Derivatives allow us to solve real-world problems involving rates of change.

LEARNING OBJECTIVE

CHA-3.G

Calculate derivatives of parametric functions.
BC ONLY

ESSENTIAL KNOWLEDGE

CHA-3.G.1

Methods for calculating derivatives of real-valued functions can be extended to parametric functions. **BC ONLY**

CHA-3.G.2

For a curve defined parametrically, the value of $\frac{dy}{dx}$ at a point on the curve is the slope of the line tangent to the curve at that point. $\frac{dy}{dx}$, the slope of the line tangent to a curve defined using parametric equations, can be determined by dividing $\frac{dy}{dt}$ by $\frac{dx}{dt}$, provided $\frac{dx}{dt}$ does not equal zero. **BC ONLY**

TOPIC 9.2

Second Derivatives of Parametric Equations

SUGGESTED SKILL

Implementing Mathematical Processes

1.E

Apply appropriate mathematical rules or procedures, with and without technology.

AVAILABLE RESOURCES

- Classroom Resource > **Vectors**
- External Resource > **Davidson Next**

Required Course Content

ENDURING UNDERSTANDING

CHA-3

Derivatives allow us to solve real-world problems involving rates of change.

LEARNING OBJECTIVE

CHA-3.G

Calculate derivatives of parametric functions.

BC ONLY

ESSENTIAL KNOWLEDGE

CHA-3.G.3

$\frac{d^2y}{dx^2}$ can be calculated by dividing $\frac{d}{dt}\left(\frac{dy}{dx}\right)$ by $\frac{dx}{dt}$. **BC ONLY**

SUGGESTED SKILL

Implementing Mathematical Processes

1.D

Identify an appropriate mathematical rule or procedure based on the relationship between concepts or processes to solve problems.

AVAILABLE RESOURCES

- Classroom Resource > **Vectors**
- External Resource > **Davidson Next**

TOPIC 9.3

Finding Arc Lengths of Curves Given by Parametric Equations

Required Course Content

ENDURING UNDERSTANDING

CHA-6

Definite integrals allow us to solve problems involving the accumulation of change in length over an interval.

LEARNING OBJECTIVE

CHA-6.B

Determine the length of a curve in the plane defined by parametric functions, using a definite integral.

BC ONLY

ESSENTIAL KNOWLEDGE

CHA-6.B.1

The length of a parametrically defined curve can be calculated using a definite integral.

BC ONLY

TOPIC 9.4
Defining and Differentiating Vector-Valued Functions

Required Course Content

ENDURING UNDERSTANDING

CHA-3

Derivatives allow us to solve real-world problems involving rates of change.

LEARNING OBJECTIVE

CHA-3.H

Calculate derivatives of vector-valued functions.
BC ONLY

ESSENTIAL KNOWLEDGE

CHA-3.H.1

Methods for calculating derivatives of real-valued functions can be extended to vector-valued functions. **BC ONLY**

SUGGESTED SKILL

Implementing Mathematical Processes

1.D

Identify an appropriate mathematical rule or procedure based on the relationship between concepts or processes to solve problems.

AVAILABLE RESOURCES

- Classroom Resource > **Vectors**
- External Resource > **Davidson Next**

SUGGESTED SKILL

Implementing Mathematical Processes

1.E

Apply appropriate mathematical rules or procedures, with and without technology.

AVAILABLE RESOURCES

- Classroom Resource > **Vectors**
- External Resource > **Davidson Next**

TOPIC 9.5

Integrating Vector-Valued Functions

Required Course Content

ENDURING UNDERSTANDING

FUN-8

Solving an initial value problem allows us to determine an expression for the position of a particle moving in the plane.

LEARNING OBJECTIVE

FUN-8.A

Determine a particular solution given a rate vector and initial conditions.

BC ONLY

ESSENTIAL KNOWLEDGE

FUN-8.A.1

Methods for calculating integrals of real-valued functions can be extended to parametric or vector-valued functions. **BC ONLY**

TOPIC 9.6

Solving Motion Problems Using Parametric and Vector-Valued Functions

Required Course Content

ENDURING UNDERSTANDING

FUN-8

Solving an initial value problem allows us to determine an expression for the position of a particle moving in the plane.

LEARNING OBJECTIVE

FUN-8.B

Determine values for positions and rates of change in problems involving planar motion. **BC ONLY**

ESSENTIAL KNOWLEDGE

FUN-8.B.1

Derivatives can be used to determine velocity, speed, and acceleration for a particle moving along a curve in the plane defined using parametric or vector-valued functions.
BC ONLY

FUN-8.B.2

For a particle in planar motion over an interval of time, the definite integral of the velocity vector represents the particle's displacement (net change in position) over the interval of time, from which we might determine its position. The definite integral of speed represents the particle's total distance traveled over the interval of time. **BC ONLY**

SUGGESTED SKILL

Implementing Mathematical Processes

1.E

Apply appropriate mathematical rules or procedures, with and without technology.

AVAILABLE RESOURCES

- Classroom Resource > **Vectors**
- External Resource > **Davidson Next**

SUGGESTED SKILL

Connecting Representations

2.D

Identify how mathematical characteristics or properties of functions are related in different representations.

AVAILABLE RESOURCE

- External Resource > **Davidson Next**

TOPIC 9.7

Defining Polar Coordinates and Differentiating in Polar Form

Required Course Content

ENDURING UNDERSTANDING

FUN-3

Recognizing opportunities to apply derivative rules can simplify differentiation.

LEARNING OBJECTIVE

FUN-3.G

Calculate derivatives of functions written in polar coordinates. **BC ONLY**

ESSENTIAL KNOWLEDGE

FUN-3.G.1

Methods for calculating derivatives of real-valued functions can be extended to functions in polar coordinates. **BC ONLY**

FUN-3.G.2

For a curve given by a polar equation $r = f(\theta)$, derivatives of r, x, and y with respect to θ, and first and second derivatives of y with respect to x can provide information about the curve.
BC ONLY

TOPIC 9.8

Find the Area of a Polar Region or the Area Bounded by a Single Polar Curve

Required Course Content

ENDURING UNDERSTANDING

CHA-5

Definite integrals allow us to solve problems involving the accumulation of change in area or volume over an interval.

LEARNING OBJECTIVE

CHA-5.D

Calculate areas of regions defined by polar curves using definite integrals. **BC ONLY**

ESSENTIAL KNOWLEDGE

CHA-5.D.1

The concept of calculating areas in rectangular coordinates can be extended to polar coordinates. **BC ONLY**

SUGGESTED SKILL

Justification

3.D

Apply an appropriate mathematical definition, theorem, or test.

AVAILABLE RESOURCE

- External Resource > **Davidson Next**

SUGGESTED SKILL

Justification

3.D

Apply an appropriate mathematical definition, theorem, or test.

AVAILABLE RESOURCE

- External Resource > **Davidson Next**

TOPIC 9.9

Finding the Area of the Region Bounded by Two Polar Curves

Required Course Content

ENDURING UNDERSTANDING

CHA-5

Definite integrals allow us to solve problems involving the accumulation of change in area or volume over an interval.

LEARNING OBJECTIVE

CHA-5.D

Calculate areas of regions defined by polar curves using definite integrals. **BC ONLY**

ESSENTIAL KNOWLEDGE

CHA-5.D.2

Areas of regions bounded by polar curves can be calculated with definite integrals. **BC ONLY**

AP CALCULUS AB AND BC

UNIT 10 BC ONLY

Infinite Sequences and Series

AP EXAM WEIGHTING **17–18%** BC ONLY

CLASS PERIODS **~17–18**

Remember to go to **AP Classroom** to assign students the online **Personal Progress Check** for this unit.

Whether assigned as homework or completed in class, the **Personal Progress Check** provides each student with immediate feedback related to this unit's topics and skills.

Personal Progress Check 10

Multiple-choice: ~45 questions
Free-response: 3 questions

UNIT 10

AP EXAM WEIGHTING	17–18% BC
CLASS PERIODS	~17–18 BC

Infinite Sequences and Series

BIG IDEA 1
Limits **LIM**

- How can the sum of infinitely many discrete terms be a finite value or represent a continuous function?

Developing Understanding

In this unit, students need to understand that a sum of infinitely many terms may converge to a finite value. They can develop intuition by exploring graphs, tables, and symbolic expressions for series that converge and diverge and for Taylor polynomials. Students should build connections to past learning, such as how evaluating improper integrals relates to the integral test or how using limiting cases of power series to represent continuous functions relates to differentiation and integration. Students who rely solely on memorizing a long list of tests and procedures typically find little success achieving a lasting conceptual understanding of series.

Building the Mathematical Practices

1.E 1.F 2.C 3.D

In Unit 10, students will need to develop proficiency with complex series notation and the ability to communicate their reasoning. Emphasize appropriate use of notation, precision of language, and establishing conditions for using a particular test. Remind students that a sound justification relies upon both mathematical evidence and reasons why that evidence supports the conclusion.

Additionally, students will need to practice determining which application is appropriate for different scenarios (for example, using the definitions of harmonic or p-series to classify certain infinite series) and then applying associated procedures accurately. Students will also need to practice using Taylor polynomials to approximate the value of a function, choosing and implementing an appropriate method to bound the error involved in the approximation, and effectively communicating supporting work.

Connecting representations is an important skill to develop in this unit. For example, students will need to identify infinite power series to represent functions presented symbolically or move between graphic and symbolic representations of an interval of convergence.

Preparing for the AP Exam

Students are more likely to demonstrate an incomplete understanding of series or to struggle with communicating their understanding of it compared to other topics. Continue to model and expect correct notation and language to present solutions, explain reasoning, and justify conclusions. For example, using the ratio test to find a radius of convergence, or operating on a known series to create another series, requires proficient, well-presented algebra. Applying a convergence test requires explicit verification that all necessary conditions are met. Determining that a given number is an error bound requires calculating an appropriate value and communicating that the value is less than the given number.

Intentional focus on the recurrent theme of using limiting cases to move from discrete approximations to analytic calculations and determinations is one way to help students to finish the year with a strong performance on the AP Exam and to come away with an enduring, meaningful understanding of calculus.

UNIT AT A GLANCE

Enduring Understanding	Topic	Suggested Skills	Class Periods ~17–18 CLASS PERIODS
LIM-7	**10.1 Defining Convergent and Divergent Infinite Series**	3.D Apply an appropriate mathematical definition, theorem, or test.	
	10.2 Working with Geometric Series	3.D Apply an appropriate mathematical definition, theorem, or test.	
	10.3 The *n*th Term Test for Divergence	3.D Apply an appropriate mathematical definition, theorem, or test.	
	10.4 Integral Test for Convergence	3.D Apply an appropriate mathematical definition, theorem, or test.	
	10.5 Harmonic Series and *p*-Series	3.B Identify an appropriate mathematical definition, theorem, or test to apply.	
	10.6 Comparison Tests for Convergence	3.D Apply an appropriate mathematical definition, theorem, or test.	
	10.7 Alternating Series Test for Convergence	3.D Apply an appropriate mathematical definition, theorem, or test.	
	10.8 Ratio Test for Convergence	3.D Apply an appropriate mathematical definition, theorem, or test.	
	10.9 Determining Absolute or Conditional Convergence	3.D Apply an appropriate mathematical definition, theorem, or test.	
	10.10 Alternating Series Error Bound	1.E Apply appropriate mathematical rules or procedures, with and without technology.	

continued on next page

UNIT AT A GLANCE *(cont'd)*

Enduring Understanding	Topic	Suggested Skills	Class Periods ~17–18 CLASS PERIODS
LIM-8	**10.11 Finding Taylor Polynomial Approximations of Functions**	3.D Apply an appropriate mathematical definition, theorem, or test. 2.C Identify a re-expression of mathematical information presented in a given representation.	
	10.12 Lagrange Error Bound	1.F Explain how an approximated value relates to the actual value.	
	10.13 Radius and Interval of Convergence of Power Series	2.C Identify a re-expression of mathematical information presented in a given representation.	
	10.14 Finding Taylor or Maclaurin Series for a Function	2.C Identify a re-expression of mathematical information presented in a given representation.	
	10.15 Representing Functions as Power Series	3.D Apply an appropriate mathematical definition, theorem, or test.	

Go to **AP Classroom** to assign the **Personal Progress Check** for Unit 10.
Review the results in class to identify and address any student misunderstandings.

SAMPLE INSTRUCTIONAL ACTIVITIES

The sample activities on this page are optional and are offered to provide possible ways to incorporate various instructional approaches into the classroom. Teachers do not need to use these activities or instructional approaches and are free to alter or edit them. The examples below were developed in partnership with teachers from the AP community to share ways that they approach teaching some of the topics in this unit. Please refer to the Instructional Approaches section beginning on p. 199 for more examples of activities and strategies.

Activity	Topic	Sample Activity
1	10.1	**Predict and Confirm** Demonstrate a geometric series, the harmonic series, and the alternating series by distributing pieces of a donut, pizza, or licorice. Ask the class to predict how much the student(s) will eventually receive in total. For instance, give three students each one-fourth, then one-fourth of the remaining fourth, and so on. Students should guess that since the remaining part is approaching zero, each student will eventually receive one-third. For alternating harmonic series, give one student a whole piece, then take away ½, then give ⅓, then take away ¼, and so forth.
2	10.2 10.3 10.4 10.5 10.6 10.7 10.8	**Graphic Organizer** Put students in groups with poster paper and have them organize and explain all the series tests using pictures, text, flowcharts, cartoons, or other drawings. Have them include each test's conditions and how to choose which test to apply.
3	10.13	**Odd One Out** Begin by modeling an example, such as three images that are pieces of furniture and one image of a houseplant, explaining that the houseplant is the "odd one out" because it's not like the other images. Then distribute a set of series such that all of the series, except one, have something in common. For example, all of the series except one could have the same type of interval of convergence (open, closed, open at left or right endpoint). Or they could converge only at the series' center or they could converge for all real numbers. Then have students decide which series in the set is the "odd one out."
4	10.15	**Collaborative Poster** Use this strategy for groups to create their own free-response questions. Give each group of four students a basic series, such as $\sin(x)$, $\cos(x)$, e^x, or $\frac{1}{1+x}$. Ask the first two members to choose a manipulation of the series and show work to complete the task. Ask the final two members to watch silently and confirm the first two steps. Then the final two members choose further actions to perform on the new series (i.e., differentiate, integrate, and find interval of convergence).

TOPIC 10.1

Defining Convergent and Divergent Infinite Series

Required Course Content

ENDURING UNDERSTANDING

LIM-7

Applying limits may allow us to determine the finite sum of infinitely many terms.

LEARNING OBJECTIVE

LIM-7.A

Determine whether a series converges or diverges.

BC ONLY

ESSENTIAL KNOWLEDGE

LIM-7.A.1

The nth partial sum is defined as the sum of the first n terms of a series. **BC ONLY**

LIM-7.A.2

An infinite series of numbers converges to a real number S (or has sum S), if and only if the limit of its sequence of partial sums exists and equals S. **BC ONLY**

SUGGESTED SKILL

Justification

3.D

Apply an appropriate mathematical definition, theorem, or test.

AVAILABLE RESOURCES

- Classroom Resource > **Infinite Series**
- External Resource > **Davidson Next**

SUGGESTED SKILL

Justification

3.D
Apply an appropriate mathematical definition, theorem, or test.

AVAILABLE RESOURCES

- Classroom Resource > **Infinite Series**
- External Resource > **Davidson Next**

TOPIC 10.2
Working with Geometric Series

Required Course Content

ENDURING UNDERSTANDING

LIM-7
Applying limits may allow us to determine the finite sum of infinitely many terms.

LEARNING OBJECTIVE

LIM-7.A
Determine whether a series converges or diverges.
BC ONLY

ESSENTIAL KNOWLEDGE

LIM-7.A.3
A geometric series is a series with a constant ratio between successive terms. **BC ONLY**

LIM-7.A.4
If a is a real number and r is a real number such that $|r| < 1$, then the geometric series $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$. **BC ONLY**

TOPIC 10.3

The *n*th Term Test for Divergence

Required Course Content

ENDURING UNDERSTANDING

LIM-7

Applying limits may allow us to determine the finite sum of infinitely many terms.

LEARNING OBJECTIVE

LIM-7.A

Determine whether a series converges or diverges.

BC ONLY

ESSENTIAL KNOWLEDGE

LIM-7.A.5

The nth term test is a test for divergence of a series. **BC ONLY**

SUGGESTED SKILL

Justification

3.D

Apply an appropriate mathematical definition, theorem, or test.

AVAILABLE RESOURCES

- Classroom Resource > **Infinite Series**
- External Resource > **Davidson Next**

SUGGESTED SKILL

Justification

3.D

Apply an appropriate mathematical definition, theorem, or test.

AVAILABLE RESOURCES

- Classroom Resource > **Infinite Series**
- External Resource > **Davidson Next**

TOPIC 10.4

Integral Test for Convergence

Required Course Content

ENDURING UNDERSTANDING

LIM-7

Applying limits may allow us to determine the finite sum of infinitely many terms.

LEARNING OBJECTIVE

LIM-7.A

Determine whether a series converges or diverges.

BC ONLY

ESSENTIAL KNOWLEDGE

LIM-7.A.6

The integral test is a method to determine whether a series converges or diverges.

BC ONLY

TOPIC 10.5

Harmonic Series and p-Series

SUGGESTED SKILL

Justification

3.B

Identify an appropriate mathematical definition, theorem, or test to apply.

AVAILABLE RESOURCES

- Classroom Resource > **Infinite Series**
- External Resource > **Davidson Next**

Required Course Content

ENDURING UNDERSTANDING

LIM-7

Applying limits may allow us to determine the finite sum of infinitely many terms.

LEARNING OBJECTIVE

LIM-7.A

Determine whether a series converges or diverges.

BC ONLY

ESSENTIAL KNOWLEDGE

LIM-7.A.7

In addition to geometric series, common series of numbers include the harmonic series, the alternating harmonic series, and p-series.

BC ONLY

SUGGESTED SKILL

Justification

3.D

Apply an appropriate mathematical definition, theorem, or test.

AVAILABLE RESOURCES

- Classroom Resource > **Infinite Series**
- External Resource > **Davidson Next**

TOPIC 10.6

Comparison Tests for Convergence

Required Course Content

ENDURING UNDERSTANDING

LIM-7

Applying limits may allow us to determine the finite sum of infinitely many terms.

LEARNING OBJECTIVE

LIM-7.A

Determine whether a series converges or diverges.

BC ONLY

ESSENTIAL KNOWLEDGE

LIM-7.A.8

The comparison test is a method to determine whether a series converges or diverges.

BC ONLY

LIM-7.A.9

The limit comparison test is a method to determine whether a series converges or diverges. **BC ONLY**

TOPIC 10.7
Alternating Series Test for Convergence

Required Course Content

ENDURING UNDERSTANDING

LIM-7

Applying limits may allow us to determine the finite sum of infinitely many terms.

LEARNING OBJECTIVE

LIM-7.A

Determine whether a series converges or diverges.
BC ONLY

ESSENTIAL KNOWLEDGE

LIM-7.A.10

The alternating series test is a method to determine whether an alternating series converges. **BC ONLY**

SUGGESTED SKILL

Justification

3.D

Apply an appropriate mathematical definition, theorem, or test.

AVAILABLE RESOURCES

- Classroom Resource > **Infinite Series**
- The Exam > **Commentary on the Instructions for the Free Response Section of the AP Calculus Exams**
- External Resource > **Davidson Next**

SUGGESTED SKILL

Justification

3.D
Apply an appropriate mathematical definition, theorem, or test.

AVAILABLE RESOURCES

- Classroom Resource > **Infinite Series**
- External Resource > **Davidson Next**

TOPIC 10.8
Ratio Test for Convergence

Required Course Content

ENDURING UNDERSTANDING

LIM-7
Applying limits may allow us to determine the finite sum of infinitely many terms.

LEARNING OBJECTIVE

LIM-7.A
Determine whether a series converges or diverges.
BC ONLY

ESSENTIAL KNOWLEDGE

LIM-7.A.11
The ratio test is a method to determine whether a series of numbers converges or diverges. **BC ONLY**

EXCLUSION STATEMENT
The nth term test for divergence, and the integral test, comparison test, limit comparison test, alternating series test, and ratio test for convergence are assessed on the AP Calculus BC Exam. Other methods are not assessed on the exam. However, teachers may include additional methods in the course, if time permits.

TOPIC 10.9

Determining Absolute or Conditional Convergence

Required Course Content

ENDURING UNDERSTANDING

LIM-7

Applying limits may allow us to determine the finite sum of infinitely many terms.

LEARNING OBJECTIVE

LIM-7.A

Determine whether a series converges or diverges.

BC ONLY

ESSENTIAL KNOWLEDGE

LIM-7.A.12

A series may be absolutely convergent, conditionally convergent, or divergent.

BC ONLY

LIM-7.A.13

If a series converges absolutely, then it converges. **BC ONLY**

LIM-7.A.14

If a series converges absolutely, then any series obtained from it by regrouping or rearranging the terms has the same value.

BC ONLY

SUGGESTED SKILL

Justification

3.D

Apply an appropriate mathematical definition, theorem, or test.

AVAILABLE RESOURCES

- Classroom Resource > **Infinite Series**
- External Resource > **Davidson Next**

SUGGESTED SKILL

Implementing Mathematical Processes

1.E

Apply appropriate mathematical rules or procedures, with and without technology.

AVAILABLE RESOURCES

- Classroom Resource > **Infinite Series**
- External Resource > **Davidson Next**

TOPIC 10.10

Alternating Series Error Bound

Required Course Content

ENDURING UNDERSTANDING

LIM-7

Applying limits may allow us to determine the finite sum of infinitely many terms.

LEARNING OBJECTIVE

LIM-7.B

Approximate the sum of a series. **BC ONLY**

ESSENTIAL KNOWLEDGE

LIM-7.B.1

If an alternating series converges by the alternating series test, then the alternating series error bound can be used to bound how far a partial sum is from the value of the infinite series. **BC ONLY**

TOPIC 10.11

Finding Taylor Polynomial Approximations of Functions

SUGGESTED SKILLS

Justification

3.D
Apply an appropriate mathematical definition, theorem, or test.

Connecting Representations

2.C
Identify a re-expression of mathematical information presented in a given representation.

AVAILABLE RESOURCES

- Classroom Resource > **Infinite Series**
- External Resource > **Davidson Next**

Required Course Content

ENDURING UNDERSTANDING

LIM-8

Power series allow us to represent associated functions on an appropriate interval.

LEARNING OBJECTIVE

LIM-8.A

Represent a function at a point as a Taylor polynomial.
BC ONLY

ESSENTIAL KNOWLEDGE

LIM-8.A.1

The coefficient of the nth degree term in a Taylor polynomial for a function f centered at $x = a$ is $\frac{f^{(n)}(a)}{n!}$. **BC ONLY**

LIM-8.A.2

In many cases, as the degree of a Taylor polynomial increases, the nth degree polynomial will approach the original function over some interval. **BC ONLY**

LEARNING OBJECTIVE

LIM-8.B

Approximate function values using a Taylor polynomial. **BC ONLY**

ESSENTIAL KNOWLEDGE

LIM-8.B.1

Taylor polynomials for a function f centered at $x = a$ can be used to approximate function values of f near $x = a$. **BC ONLY**

SUGGESTED SKILL

Implementing Mathematical Processes

1.F

Explain how an approximated value relates to the actual value.

AVAILABLE RESOURCES

- Classroom Resource > **Infinite Series**
- Classroom Resource > **Approximation**
- External Resource > **Davidson Next**

TOPIC 10.12
Lagrange Error Bound

Required Course Content

ENDURING UNDERSTANDING

LIM-8

Power series allow us to represent associated functions on an appropriate interval.

LEARNING OBJECTIVE

LIM-8.C

Determine the error bound associated with a Taylor polynomial approximation.
BC ONLY

ESSENTIAL KNOWLEDGE

LIM-8.C.1

The Lagrange error bound can be used to determine a maximum interval for the error of a Taylor polynomial approximation to a function.
BC ONLY

LIM-8.C.2

In some situations, the alternating series error bound can be used to bound the error of a Taylor polynomial approximation to the value of a function. **BC ONLY**

TOPIC 10.13

Radius and Interval of Convergence of Power Series

Required Course Content

ENDURING UNDERSTANDING

LIM-8

Power series allow us to represent associated functions on an appropriate interval.

LEARNING OBJECTIVE

LIM-8.D

Determine the radius of convergence and interval of convergence for a power series. **BC ONLY**

ESSENTIAL KNOWLEDGE

LIM-8.D.1

A power series is a series of the form $\sum_{n=0}^{\infty} a_n (x-r)^n$, where n is a non-negative integer, $\{a_n\}$ is a sequence of real numbers, and r is a real number. **BC ONLY**

LIM-8.D.2

If a power series converges, it either converges at a single point or has an interval of convergence. **BC ONLY**

LIM-8.D.3

The ratio test can be used to determine the radius of convergence of a power series. **BC ONLY**

LIM-8.D.4

The radius of convergence of a power series can be used to identify an open interval on which the series converges, but it is necessary to test both endpoints of the interval to determine the interval of convergence. **BC ONLY**

LIM-8.D.5

If a power series has a positive radius of convergence, then the power series is the Taylor series of the function to which it converges over the open interval. **BC ONLY**

LIM-8.D.6

The radius of convergence of a power series obtained by term-by-term differentiation or term-by-term integration is the same as the radius of convergence of the original power series. **BC ONLY**

SUGGESTED SKILL

Connecting Representations

2.C

Identify a re-expression of mathematical information presented in a given representation.

AVAILABLE RESOURCES

- Classroom Resource > **Infinite Series**
- External Resource > **Davidson Next**

SUGGESTED SKILL

Connecting Representations

2.C

Identify a re-expression of mathematical information presented in a given representation.

AVAILABLE RESOURCES

- Classroom Resource > **Infinite Series**
- AP Online Teacher Community Discussion > **Question on Taylor Polynomials**
- External Resource > **Davidson Next**

TOPIC 10.14

Finding Taylor or Maclaurin Series for a Function

Required Course Content

ENDURING UNDERSTANDING

LIM-8

Power series allow us to represent associated functions on an appropriate interval.

LEARNING OBJECTIVE	ESSENTIAL KNOWLEDGE
LIM-8.E Represent a function as a Taylor series or a Maclaurin series. **BC ONLY**	**LIM-8.E.1** A Taylor polynomial for $f(x)$ is a partial sum of the Taylor series for $f(x)$. **BC ONLY**
LIM-8.F Interpret Taylor series and Maclaurin series. **BC ONLY**	**LIM-8.F.1** The Maclaurin series for $\frac{1}{1-x}$ is a geometric series. **BC ONLY** **LIM-8.F.2** The Maclaurin series for $\sin x$, $\cos x$, and e^x provides the foundation for constructing the Maclaurin series for other functions. **BC ONLY**

TOPIC 10.15

Representing Functions as Power Series

Required Course Content

ENDURING UNDERSTANDING

LIM-8

Power series allow us to represent associated functions on an appropriate interval.

LEARNING OBJECTIVE

LIM-8.G

Represent a given function as a power series. **BC ONLY**

ESSENTIAL KNOWLEDGE

LIM-8.G.1

Using a known series, a power series for a given function can be derived using operations such as term-by-term differentiation or term-by-term integration, and by various methods (e.g., algebraic processes, substitutions, or using properties of geometric series). **BC ONLY**

SUGGESTED SKILL

Justification

3.D

Apply an appropriate mathematical definition, theorem, or test.

AVAILABLE RESOURCES

- Classroom Resource > **Infinite Series**
- External Resource > **Davidson Next**

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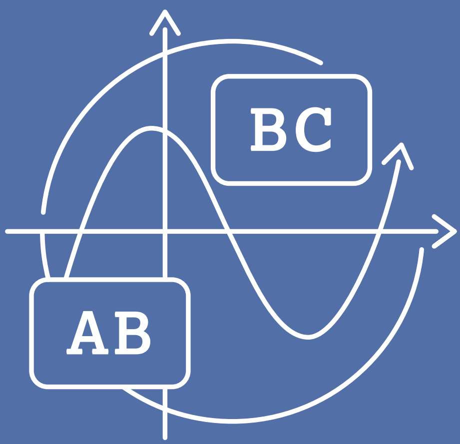
AP CALCULUS AB AND BC

Instructional Approaches



BC
AB

Selecting and Using Course Materials

Textbooks

While College Board provides examples of textbooks to help determine whether a text is considered appropriate in meeting the AP Calculus Course Audit curricular requirement, teachers select textbooks locally. AP Central has a **list of textbook examples** that meet the resource requirements.

Graphing Calculators and Other Technologies in AP Calculus

The use of a graphing calculator is considered an integral part of the AP Calculus courses, and it is required on some portions of the exams. Professional mathematics organizations such as the National Council of Teachers of Mathematics (NCTM), the Mathematical Association of America (MAA), and the National Academy of Sciences (NAS) Board on Mathematical Sciences and Their Applications have strongly endorsed the use of calculators in mathematics instruction and testing.

Graphing calculators are valuable tools for achieving multiple components of the mathematical practices, including using technology to develop conjectures, connecting concepts to their visual representations, solving problems, and critically interpreting and accurately reporting information. The AP Calculus courses also support the use of other technologies that are available to students and encourage teachers to incorporate technology into instruction in a variety of ways as a means of facilitating discovery and reflection.

Appropriate examples of graphing calculator use in AP Calculus include but certainly are not limited to:

- Zooming to reveal local linearity
- Constructing a table of values to conjecture a limit
- Developing a visual representation of Riemann sums approaching a definite integral
- Graphing Taylor polynomials to understand intervals of convergence for Taylor series
- Drawing a slope field and investigating how the choice of initial condition affects the solution to a differential equation

Online Tools and Resources

In addition to the resources offered through a teacher's choice of textbook, many useful educational resources exist online—and are often free. Below are a few examples of these types of resources. This is not a comprehensive list, nor is it an endorsement of any of these resources. Teachers can sample different resources to find which ones can most benefit students.

- **Classpad (Casio)**
- **Desmos**

- **Geogebra**
- **Hewlett Packard Educational Resources**
- **Math Open Reference**
- **Texas Instruments Education**
- **TI in Focus: AP Calculus**
- **Visual Calculus**
- **Wolfram Math World**

Professional Organizations

Professional organizations also serve as excellent resources for best practices and professional development opportunities. Following is a list of prominent organizations that serve the math education communities.

Mathematical Association of America
maa.org
The MAA is the world's largest community of mathematicians, students, and enthusiasts. The organization's mission is to advance the understanding of mathematics and its impact on our world.

National Math and Science Initiative (NMSI)
nms.org
NMSI is a nonprofit organization whose mission is to improve student performance in the subjects of science, technology, engineering, and math (STEM) in the United States. They employ experienced AP teachers to train students and teachers in the STEM courses.

National Council of Teachers of Mathematics (NCTM)
nctm.org
Founded in 1920, NCTM is the world's largest mathematics education organization, advocating for high-quality mathematics teaching and learning for all students.

Instructional Strategies

The AP Calculus course framework outlines the concepts and skills students must master in order to be successful on the AP Exam. In order to address those concepts and skills effectively, it helps to incorporate a variety of instructional approaches and best practices into daily lessons and activities. The following table presents strategies that can help students apply their understanding of course concepts.

Strategy	Definition	Purpose	Example
Always, Sometimes, Never	Students are given justification statements and are asked to determine whether that line of reasoning is always true, sometimes true, or never true.	Allows students to hone the precision and thoroughness of their justifications by having them determine the validity of given statements, and by explaining why certain statements could be true under certain conditions.	When justifying properties and behaviors of functions based on derivatives, provide students with a list of statements such as the following (assuming $f(x)$ is twice differentiable): ▪ If $f'(c)=0$ and $f''(c)<0$, then $x=c$ _____ locates a local maximum of $f(x)$. ▪ If $f'(x)$ changes from negative to positive at $x=c$, then $x=c$ _____ locates a relative maximum of $f(x)$. ▪ If $f'(c)=0$ and $f''(c)>0$, then $x=c$ _____ locates an absolute minimum of $f(x)$. Then have students write "always," "sometimes," or "never" in the appropriate blanks to make the statement true. (Solutions for the sample statements above, respectively: always, never, sometimes)
Ask the expert	Students are assigned as "experts" on problems they have mastered; groups rotate through the expert stations to learn about problems they have not yet mastered.	Provides opportunities for students to share their knowledge and learn from one another.	When learning rules of differentiation, assign students as "experts" on product rule, quotient rule, chain rule, and derivatives of transcendental functions. Students rotate through stations in groups, working with the station expert to complete a series of problems using the corresponding rule.

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Strategy	Definition	Purpose	Example
Collaborative Poster	In groups of four, students are assigned a colored marker and a subpart of a free-response question. Each student completes their subpart on the poster using their assigned marker.	Allows students time to analyze each other's work and explain their reasoning to others.	Have students divide their poster into four corners and give each student a colored marker for their part of the free-response question: ▪ Green for part a ▪ Purple for part b ▪ Red for part c ▪ Blue for part d After completing their parts, have students hang their posters in the room and walk around to review each other's work. Students can use yellow sticky notes to leave positive comments and pink sticky notes for recommended corrections.
Concepts with Color	Students use colored highlighters, post-its, or sticky flags to note where particular concepts, vocabulary, or processes appear or to note connections between different representations of the same concept.	Useful for illustrating, emphasizing, and/or tracing how a particular concept manifests in a problem, process, or verbal description. Can be modified for students with color blindness such that symbols or typeface features are used instead of color.	At the end of the course, have students revisit their notes from previous units and use highlighters or sticky notes of different colors to tag places where specific concepts have been discussed in different contexts, for example: ▪ Blue for "continuity" ▪ Orange for "rate of change" ▪ Green for "accumulation" Then have students discuss in their groups the themes and key aspects of those concepts as they have threaded throughout the course.
Construct an Argument	Students use mathematical reasoning to present assumptions about mathematical situations, support conjectures with mathematically relevant and accurate data, and provide a logical progression of ideas leading to a conclusion that makes sense.	Helps develop the process of evaluating mathematical information, developing reasoning skills, and enhancing communication skills in supporting conjectures and conclusions.	This strategy can be used with word problems that do not lend themselves to immediate application of a formula or mathematical process. Provide distance and velocity graphs that represent a motorist's behavior through several towns on a map and ask students to construct a mathematical argument either in defense of or against a police officer's charge of speeding, given a known speed limit.

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Strategy	Definition	Purpose	Example
Create a Plan	Students analyze the tasks in a problem and create a process for completing the tasks by finding the information needed, interpreting data, choosing how to solve a problem, communicating the results, and verifying accuracy.	Assists in breaking tasks into smaller parts and identifying the steps needed to complete the entire task.	Given an optimization problem that asks for a choice between two boxes with different dimensions but the same cross-sectional perimeter, have students identify the steps needed to determine which box will hold the most candy. This involves selecting an appropriate formula, differentiating the resulting function, applying the second derivative test, and interpreting the results.
Create Representations	Students create pictures, tables, graphs, lists, equations, models, and/or verbal expressions to interpret text or data.	Helps organize information using multiple ways to present data and to answer a question or show a problem solution.	In order to evaluate limits, introduce a variety of methods, including constructing a graph, creating a table, directly substituting a given value into the function, or applying an algebraic process.
Critique Reasoning	Through collaborative discussion, students respond to the arguments of others and question the use of mathematical terminology, assumptions, and conjectures to improve understanding and justify and communicate conclusions.	Helps students learn from each other as they make connections between mathematical concepts and learn to verbalize their understanding and support their arguments with reasoning and data that make sense to peers.	Given a table that lists a jogger's velocity at five different times during his or her workout, have students explain the meaning of the definite integral of the absolute value of the velocity function between the first and the last time recorded. As students discuss their responses in groups, they learn how to communicate specific concepts and quantities using mathematical notation and terminology.
Debriefing	Students discuss their understanding of a concept to lead to a consensus on its meaning.	Helps clarify misconceptions and deepen understanding of content.	In order to discern the difference between average rate of change and instantaneous rate of change, have students roll a ball down a simplified ramp and measure the distance the ball travels over time, every second for five seconds. Plotting the points and sketching a curve of best fit, have students discuss how they might determine the average velocity of the ball over the five seconds and then the instantaneous velocity of the ball at three seconds. A discussion in which students address the distinction between the ball's velocity between two points and its velocity at a single particular time would assist in clarifying the concept and mathematical process of arriving at the correct answers.

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Strategy	Definition	Purpose	Example
Discussion Groups	Students work within groups to discuss content, create problem solutions, and explain and justify a solution.	Aids understanding through the sharing of ideas, interpretation of concepts, and analysis of problem scenarios.	Once students learn all methods of integration and choose which is the most appropriate given a particular function, have them discuss in small groups, with pencils down, why a specific method should be used over another.
Distractor Analysis	Students examine answers to a multiple-choice question and determine which answers are incorrect and why.	Allows students to become familiar with the procedural or communication errors that might cause them to get a problem wrong.	To help students practice applying derivative rules, provide multiple-choice questions with a set of "distractor answers," along with a separate list of descriptions—one for each answer choice—without telling students which description applies to which answer. For example, if the question asks for the derivative of $f(x)=(x-1)(x^2+2)^3$, then one distractor answer could be $6x(x-1)(x^2+2)^2$, and its corresponding description could be, "This choice is for the student who applied the chain rule correctly to $(x^2+2)^3$ but who never applied the product rule." Have students match up each distractor answer choice to its corresponding description in the list.
Error Analysis	Students analyze an existing solution to determine whether (or where) errors have occurred.	Allows students to troubleshoot existing errors so they can apply procedures correctly when they do the same types of problems on their own.	When students begin to evaluate definite integrals, have them analyze their answers and troubleshoot any errors that might lead to a negative area when there is a positive accumulation.
Four Corners	Students are given a sheet of paper that's been divided vertically and horizontally with four equal sections and a topic name in the middle. Each section has a problem that uses a different representation (e.g., numerical, graphical, analytical, and verbal).	Helps to deepen understanding of a concept by having students explore different representations and make connections between those representations.	To help students learn about limits, provide a four corners activity sheet that asks them to find limits based on analytical functions in one corner, a graph in another corner, a table in another corner, and a verbal description in the last corner.

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Strategy	Definition	Purpose	Example
Graph and Switch	Students generate a graph (or sketch of a graph) to model a certain function, and then switch calculators (or papers) to review each other's solutions.	Allows students to practice creating different representations of functions, and then give and receive feedback on each other's work.	As students learn about integration and finding the area under a curve, have them use calculators to shade in the appropriate area between lower and upper limits while calculating the total accumulation. Since input keystrokes are critical in obtaining the correct numerical value, have students calculate their own answers, share their steps with a partner, and receive feedback on their calculator notation and final answer.
Graphic Organizer	Students represent ideas and information visually (e.g., Venn diagrams, flowcharts, etc.).	Provides students a visual system for organizing multiple ideas and details related to a particular concept.	In order to determine the location of relative extrema for a function, have students construct a sign chart or number line while applying the first derivative test, marking where the first derivative is positive or negative and determining where the original function is increasing or decreasing.
Guess and Check	Students guess the solution to a problem, and then check that the guess fits the information in the problem and is an accurate solution.	Allows exploration of different ways to solve a problem; may be used when other strategies for solving are not obvious.	Encourage students to employ this strategy for drawing a graphical representation of a given function, given written slope statements and/or limit notation. For example, given a statement describing the graph of a particular function, have students sketch the graph described and then check it against the solution graph.
Identify a Subtask	Students break a problem into smaller pieces whose combined outcomes lead to a solution.	Helps to organize the pieces of a complex problem and reach a complete solution.	After providing students with the rates in which rainwater flows into and out of a drainpipe, ask them to find how many cubic feet of water flow into it during a specific time period and whether the amount of water in the pipe is increasing or decreasing at a particular instant. Students would begin by distinguishing functions from one another and determining whether differentiation or integration is appropriate. They would then perform the relevant calculations and verify whether they have answered the question.

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Strategy	Definition	Purpose	Example
Look for a Pattern	Students observe information or create visual representations to find a trend.	Helps to identify patterns that may be used to make predictions.	Patterns can be detected when approximating area under a curve using Riemann sums. Have students calculate areas using left and right endpoint rectangles, midpoint rectangles, and trapezoids, increasing and decreasing the width in order to determine the best method for approximation.
Marking the Text	Students highlight, underline, and/or annotate text to focus on key information to help understand the text or solve the problem.	Helps to identify important information in the text and make notes about the interpretation of tasks required and concepts to apply to reach a solution.	This strategy can be used with problems that involve related rates. Have students read through a given problem, underline the given static and changing quantities, list these quantities, and then use them to label a sketch that models the situation given in the problem. Then have students use this information to substitute for variables in a differential equation.
Match Mine	Students work in pairs to describe the layout of cards on their game board, without being able to see each other's boards.	Emphasizes the use of appropriate vocabulary and allows students to practice accurate communication around different representations.	Create a blank 3×3 grid as a "game board" and nine graph images depicting concepts related to limits/continuity. Pair up students and insert a folder between the partners so they can't see each other's boards. Distribute a blank grid and a set of the images to each partner. Student A arranges the graphs however they choose on the grid and then describes that arrangement so that Student B can attempt to match it on their own board. When the pair thinks that they have correctly made all matches, have them compare their arrangements to see how well they did.
Model Questions	Students answer questions from released AP Calculus Exams.	Provides challenging practice and assesses students' ability to apply multiple mathematical practices on content presented as either a multiple-choice or a free-response question.	After learning how to construct slope fields, have students practice by completing free-response questions in which they are asked to sketch slope fields for given differential equations at points indicated.

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Strategy	Definition	Purpose	Example
Notation Read Aloud	Students read symbols and notational representations aloud.	Helps students to accurately interpret symbolic representations.	This strategy can be used to introduce new symbols and mathematical notation to ensure that students learn proper terminology from the start. For example, after introducing summation notation, ask students to write or say aloud the verbal translation of a given sum.
Note-Taking	Students create a record of information while reading a text or listening to a speaker.	Helps in organizing ideas and processing information.	Have students write down verbal descriptions of the steps needed to solve a differential equation so that a record of the process can be referred to at a later point in time.
Numbered Heads Together	Students are put into groups and assigned a number (1 through 4). Members of a group work together to agree on an answer. The teacher randomly selects one number. The student with that number answers for the group.	Allows students to process information individually, and then sync with their peers to develop a common understanding of a particular concept or approach.	In groups, have students begin by solving a problem individually. When all students have finished, have them stand up and discuss their answers. Students sit back down when they have all agreed on a solution. Then call on one number in the group (1, 2, 3, or 4) and have that student stand up to share the group's answer.
Odd One Out	In groups of four, students are given four problems, images, or graphs. Three of the items should have something in common. Each student in the group works with one of the four items and must decide individually whether their item fits with the other three items and then write a reason why. Students then share their responses within their groups.	Allows students to explore relationships and underlying principles for mathematical concepts that are similar to, or different from, one another.	Begin by modeling an example, such as three images that are four-legged animals and one image that's an object, explaining why the object is the "odd one out." Then provide each student in the group with one of four images displaying graphs. Three of the graphs should represent functions with discontinuities, while the fourth graph represents a continuous function. Have students examine their graph and write on mini whiteboards, "My graph is in because . . ." or "My graph is out because . . . ." Each group discusses and signals when they've reached consensus. Then reveal the answer and explain what the three similar graphs had in common, and why the other was the odd one out.

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Strategy	Definition	Purpose	Example
Paraphrasing	Students restate in their own words the essential information in a text or problem description.	Assists with comprehension, recall of information, and problem solving.	After reading a mathematical definition from a textbook, have students express the definition in their own words. For example, with parametric equations, have students explain the difference between y being a function of x directly, and x and y both being functions of a parameter t. **BC ONLY**
Password-Style Games	Students are in pairs with one student facing the teacher and the other with their back to the teacher. The teacher projects a word and the student facing the teacher describes the word to their partner. This repeats every 10 seconds for 6 words. Students then switch roles.	Reinforces understanding of key vocabulary terms by having students use different approaches for describing those terms.	Students arrange themselves into partners (or triads), with one partner seated with their back to the teacher. When the teacher projects the term, "tangent line," the students facing the teacher provide clues to try to get their partner to write the term on their mini whiteboard. After 10 seconds, the teacher projects a new term, "derivative." The process continues for as many vocabulary terms as there are for that activity.
Predict and Confirm	Students make conjectures about what results will develop in an activity and confirm or modify the conjectures based on outcomes.	Stimulates thinking by making, checking, and correcting predictions based on evidence from the outcome.	Given two sets of cards with functions and the graphs of their derivatives, have students attempt to match the functions with their appropriate derivative match. Students then calculate the derivatives of the functions using specific rules and graph the derivatives using calculators to confirm their original match selection.
Quickwrite	Students write for a short, specific amount of time about a designated topic.	Helps generate ideas in a short time.	To help synthesize concepts after having learned how to calculate the derivative of a function at a point, have students list as many real-world situations as possible in which knowing the instantaneous rate of change of a function is advantageous.

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Strategy	Definition	Purpose	Example
Quiz-Quiz-Trade	Students answer a question independently, and then quiz a partner on the same question.	Allows students to talk to multiple peers and solve many problems while coaching each other and hearing different ways of solving or arriving at a solution.	Give students a card containing a question and have them write the answer on the back. Students then stand up and find a partner. One student quizzes the other, and then they reverse roles. They switch cards, find a new partner, and the process repeats.
Reciprocal Teaching	Students are divided into groups and each member of the group is assigned a role. Each student reads the problem and completes their assigned role for that problem. Students take turns sharing what they wrote with their group, while other group members take notes. Students then rotate roles and begin on another part of the problem.	Helps students gain an entry point into a problem by breaking it down. Also allows students to see multiple ways of approaching a problem as peers explain their solutions to one another.	To help students practice free-response questions, divide groups into four roles, each responsible for a different task on their activity sheet: 1. Read the problem. Identify key words (highlight or underline and list them). 2. Represent the key words with a picture. 3. Describe what needs to be done to complete that part in mathematical terms. 4. Explain any important details that must be on your paper to get full credit on the AP Exam (use prior knowledge). Then have students work through the problem one part at a time (including the question stem), rotating roles for each part.
Round Table	Students are given a worksheet with multiple problems and work individually to solve them, and then rotate after each problem to allow other group members to check their work.	Allows students the chance to analyze each other's work and coach their peers, if necessary.	In groups of four, give each student an identical paper with four different problems on it. Have students complete the first one on their paper, and then pass the paper clockwise to another member in their group. That student checks the first problem and then completes the second problem on the paper. Students rotate again and the process continues until each student has their original paper back.

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Strategy	Definition	Purpose	Example
Scavenger Hunt	Provide students with a "starter" question. Then place solution cards around the room. Each card should contain the solution to a previous problem, along with the next problem in the "scavenger hunt." Students walk around answering each new problem and finding the corresponding solution around the room.	Allows students to self-correct and analyze their own work if they get an answer they don't see posted.	When practicing the chain rule, place a card with a starter question somewhere in the classroom: "Find the derivative of $f(x)=\sin(4x)$." Place another card somewhere in the room with the solution to that card, plus another question, for example: "Solution: $f'(x)=4\cos(4x)$. Next problem: Find the derivative of $f(x)=(\sin(x))^4$." Continue posting solution cards with new problems until the final card presents a problem whose solution is on the original starter card (note that this solution would be added to the starter card above). Students begin by solving the starter question, then searching around the room for the solution they found, as this will lead them to the next problem they need to solve, until they end up at the first card they started with.
Sentence Starters	Students respond to a prompt by filling in the missing parts of a given sentence template.	Helps students practice communication skills by providing a starting point and modeling a sentence structure that would apply for a particular type of problem.	To help students practice writing conclusions based on the Intermediate Value Theorem (IVT), provide a function or table of values, along with a scenario for which IVT would be applicable. Then have students use a given sentence starter to justify that the function is continuous. For example, students could be given this scenario: "Given $f(x)=x^3-3x^2+5x-4$, verify that there is at least one root for $f(x)$." A sentence starter could then be provided to help students begin their justifications, such as "The function is continuous because . . . ."
Sharing and Responding	Communicating with another person or a small group of peers who respond to a proposed problem solution.	Gives students the opportunity to discuss their work with peers, to make suggestions for improvement to the work of others, and/or to receive appropriate and relevant feedback on their own work.	Given tax rate schedules for single taxpayers in a specific year, have students construct functions to represent the amount of tax paid by taxpayers in specific tax brackets. Then have students come together in a group to review the constructed functions, make any necessary corrections, and build and graph a single piecewise function to represent the tax rate schedule for single taxpayers for the specific year.

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Strategy	Definition	Purpose	Example
Simplify the Problem	Students use "friendlier" numbers or functions to help solve a problem.	Provides insight into the problem or the strategies needed to solve the problem.	When applying the chain rule for differentiation or u-substitution for integration, review how to proceed when there is no "inner function" before addressing composite functions.
Stand Up, Hand Up, Pair Up	Students are paired up and assigned a problem to solve. One student solves the problem aloud while the other student records the solution. The students reverse roles and repeat the steps. Together they stand up and put their hand up, and then find a new partner and work on a new problem.	Allows students to talk to multiple peers and hear multiple perspectives. Can be used to have students quickly review homework or warm up answers.	Pair up students and distribute a different problem to each pair. Determine which student in each pair will "solve" first. **Step 1:** Student A and Student B read the problem individually. **Step 2:** Student B solves the problem out loud while Student A records the solution on paper. **Step 3:** Student B signs Student A's paper. **Step 4:** Students reverse roles. **Step 5:** Students stand up, put their hand up, then they pair up with a new partner and continue.
Think Aloud	Students talk through a difficult problem by describing what the text means.	Helps in comprehending the text, understanding the components of a problem, and thinking about possible paths to a solution.	In order to determine if a series converges or diverges, have students ask themselves a series of questions out loud to identify series characteristics and corresponding tests (e.g., ratio, root, integral, and limit comparison) that are appropriate for determining convergence.
Think-Pair-Share (or Wait, Turn, and Talk)	Students think through a problem alone, pair with a partner to share ideas, and then conclude by sharing results with the class.	Enables the development of initial ideas that are then tested with a partner in preparation for revising ideas and sharing them with a larger group.	Given the equation of a discontinuous function, have students think of ways to make the function continuous and adjust the given equation to establish such continuity. Then have students pair with a partner to discuss their ideas before sharing out with the whole class.
Use Manipulatives	Students use objects to examine relationships between the information given.	Supports comprehension by providing a visual or hands-on representation of a problem or concept.	To visualize the steps necessary to find the volume of a solid with a known cross-section, have students build a physical model on a base with a standard function using foam board or weighted paper to construct several cross-sections.
Work Backward	Students trace a possible answer back through the solution process to the starting point.	Provides another way to check possible answers for accuracy.	Have students check whether they have found a correct antiderivative by differentiating their answer and comparing it to the original function.

Developing the Mathematical Practices

Throughout the course, students will develop mathematical practices that are fundamental to the discipline of calculus. Students will benefit from multiple opportunities to develop these skills in a scaffolded manner.

The tables that follow look at each of the mathematical practices and provide examples of questions for each skill, along with sample activities and strategies for incorporating that skill into the course.

Mathematical Practice 1: Determine expressions and values using mathematical procedures and rules

The table that follows provides examples of questions and instructional strategies for teaching students to successfully implement mathematical processes for different topics throughout the course.

Mathematical Practice 1: *Implementing Mathematical Processes*

Skills	Key Questions	Sample Activities	Sample Instructional Strategies
1.A: *Identify the question to be answered or problem to be solved.*	▪ What is the problem asking us to find? ▪ Does this answer the question being asked? ▪ Is this solution reasonable? How do you know?	Have students examine a set of free-response questions that have similar underlying structures (e.g., related rates problems) and underline the question being asked, without actually solving the problem.	▪ Marking the Text ▪ Model Questions ▪ Paraphrasing
1.B: *Identify key and relevant information to answer a question or solve a problem.*	▪ What information do you need? ▪ Did you use all of the information? ▪ Is there any information that was not needed?	After students have identified the question being asked, have them write down one piece of information they would need in order to answer that question and then have them switch with a partner to confirm.	▪ Sharing and Responding ▪ Think-Pair-Share

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Mathematical Practice 1: *Implementing Mathematical Processes (cont'd)*

Skills	Key Questions	Sample Activities	Sample Instructional Strategies
1.C: *Identify an appropriate mathematical rule or procedure based on the classification of a given expression (e.g., Use the chain rule to find the derivative of a composite function).*	▪ When should you use direct substitution to find a limit? ▪ What are the clues that tell you this is (or is not) a composite function?	Examine a function whose graph is somewhat misleading and find its limit algebraically. For example, $\lim_{x \to 0} \left[\frac{(100x - \cos(x))^2}{100{,}000} \right]$ can be easily found using direct substitution.	▪ Discussion Groups ▪ Sharing and Responding
1.D: *Identify an appropriate mathematical rule or procedure based on the relationship between concepts (e.g., rate of change and accumulation) or processes (e.g., differentiation and its inverse process, antidifferentiation) to solve problems.*	▪ Is this problem about a rate of change or about the accumulation of something? ▪ What units are provided? ▪ What units are appropriate for the solution?	Have students match different unit labels to the types of scenarios they might see in different problems: average value, average rate of change, instantaneous rate of change, velocity, acceleration, and so forth.	▪ Distractor Analysis ▪ Error Analysis ▪ Model Questions ▪ Scavenger Hunt
1.E: *Apply appropriate mathematical rules or procedures, with and without technology.*	▪ Have we solved a problem similar to this? ▪ What steps are needed? ▪ Does the power rule apply to every function that has an exponent? Why or why not?	Have students "fix" discontinuities in piecewise-defined functions by defining or redefining the value of the function at a discontinuity, so that it is equal to the limit of the function as x approaches the point of discontinuity.	▪ Collaborative Poster ▪ Distractor Analysis ▪ Identify a Subtask ▪ Simplify the Problem
1.F: *Explain how an approximated value relates to the actual value.*	▪ How is ___ related to ___? ▪ How can this be represented graphically?	Give students values for $f(2)$ and $f'(2)$ and ask them to approximate $f(2.1)$. Use graphs of curves and tangent lines to understand when a tangent line approximation is an overestimate or underestimate and when it provides the exact value of $f(2.1)$.	▪ Create Representations ▪ Guess and Check ▪ Sentence Starters

Mathematical Practice 2: Translate mathematical information from a single representation or across multiple representations

The table that follows provides examples of questions and instructional strategies for teaching students to connect representations successfully.

Mathematical Practice 2: *Connecting Representations*

Skills	Key Questions	Sample Activities	Sample Instructional Strategies
2.A: *Identify common underlying structures in problems involving different contextual situations.*	▪ What clues tell you what kind of problem this is? ▪ How is ___ similar to ____? ▪ How is ___ different from ___?	Have students create a T-chart with *Optimization* in one column and *Related Rates* in the other column, listing in each column the keywords that might appear in each type of problem.	▪ Concepts with Color ▪ Discussion Groups ▪ Graphic Organizer ▪ Marking the Text
2.B: *Identify mathematical information from graphical, numerical, analytical, and/or verbal representations.*	▪ What would a graph of this equation look like? ▪ How could this graph be represented as an equation? ▪ How can this situation be represented in a diagram or described in a sentence?	Have students match sets of cards so that each set contains the verbal, graphical, analytical, and numerical representation for the same function.	▪ Create Representations ▪ Graph and Switch ▪ Quickwrite
2.C: *Identify a re-expression of mathematical information presented in a given representation.*	▪ How would you read this expression aloud? ▪ Which of these limit expressions represents the graph shown?	Have students sketch the graph of a limit represented analytically, and then switch with a partner and have the partner check their work.	▪ Create Representations ▪ Notation Read Aloud ▪ Scavenger Hunt
2.D: *Identify how mathematical characteristics or properties of functions are related in different representations.*	▪ Why is ___ a more appropriate representation than ___?	Give students four cards, each using the same type of representation for a different derivative function (e.g., graphs of f', g', a', and b'). Have students identify which of the original functions is dissimilar from the others and why.	▪ Concepts with Color ▪ Debriefing ▪ Odd One Out
2.E: *Describe the relationships among different representations of functions and their derivatives.*	▪ How is ___ related to ___? ▪ What do you know about ___?	Provide students with a graph of f and give them one minute to write down everything they know about f based on that graph.	▪ Create Representations ▪ Graphic Organizer ▪ Quickwrite

Mathematical Practice 3: Justify reasoning and solutions

The table that follows provides examples of questions and instructional strategies for helping students to develop the skill of justification. Note that on the AP Exam, sign charts can be useful tools for understanding a problem, but they are not sufficient as justifications for conclusions about the behaviors of functions.

Mathematical Practice 3: *Justification*

Skills	Key Questions	Sample Activities	Sample Instructional Strategies
3.A: *Apply technology to develop claims and conjectures.*	▪ What would this look like on a graphing calculator? ▪ What patterns do you see? ▪ What is your hypothesis? ▪ What does it mean for a limit to approach infinity?	Use technology to examine the graphs of each of the illustrative examples in Essential Knowledge statement LIM-1.C.4. Use technology to explore graphs and tables of series which do and do not converge, such as the geometric series $\left(\frac{1}{2}\right)^n$ and 2^n. **BC ONLY**	▪ Graph and Switch ▪ Look for a Pattern
3.B: *Identify an appropriate mathematical definition, theorem, or test to apply.*	▪ Under what conditions . . . ? ▪ How could we test . . . ?	Have students categorize limit expressions according to whether or not L'Hospital's Rule would be applicable.	▪ Create a Plan ▪ Graphic Organizer ▪ Model Questions
3.C: *Confirm whether hypotheses or conditions of a selected definition, theorem, or test have been satisfied.*	▪ What conditions have been given? ▪ What else can we conclude from the given conditions? ▪ What would happen if . . . ?	Have students examine sample responses for IVT, EVT, and MVT problems to identify which responses appropriately address the conditions of the theorem being applied and which responses have not addressed the conditions.	▪ Distractor Analysis ▪ Error Analysis ▪ Think-Pair-Share
3.D: *Apply an appropriate mathematical definition, theorem, or test.*	▪ Show me an example that would NOT work in this context.	Have students create a decision tree (flowchart) for the ratio test, which actually determines absolute convergence or divergence, when it applies. **BC ONLY**	▪ Ask the Expert ▪ Distractor Analysis ▪ Error Analysis ▪ Graphic Organizer ▪ Reciprocal Teaching

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Mathematical Practice 3: *Justification (cont'd)*

Skills	Key Questions	Sample Activities	Sample Instructional Strategies
3.E: *Provide reasons or rationales for solutions and conclusions.*	▪ How do you know . . . ? ▪ What line of reasoning did you use to . . . ? ▪ What evidence do you have to support . . . ? ▪ What can you conclude from the evidence?	Have students complete a fill-in-the-blank template for explaining why the Mean Value Theorem does (or does not) apply for a given function on a given interval.	▪ Construct an Argument ▪ Critique Reasoning ▪ Graphic Organizer ▪ Quickwrite ▪ Sentence Starters ▪ Think Aloud
3.F: *Explain the meaning of mathematical solutions in context.*	▪ What units are appropriate? ▪ What does ___ mean?	Given a logistic differential equation and initial population, have students find the population as t approaches infinity, applying the idea of carrying capacity. **BC ONLY**	▪ Ask the Expert ▪ Error Analysis ▪ Model Questions ▪ Scavenger Hunt
3.G: *Confirm that solutions are accurate and appropriate.*	▪ Is this solution reasonable? How do you know?	Have students write the inverses of exponential and logarithmic functions and differentiate them using the chain rule (treating y as a function of x), and then have them write the general formula for derivatives of inverses.	▪ Distractor Analysis ▪ Error Analysis ▪ Numbered Heads Together ▪ Quiz-Quiz-Trade

Mathematical Practice 4: Use correct notation, language, and mathematical conventions to communicate results or solutions

The data indicate that students consistently struggle with communication and appropriate use of notation on the AP Calculus Exams. Students often need targeted support to develop these skills. Remind students that communicating reasoning is at least as important as finding a solution. Well-communicated reasoning validates solutions.

Reinforce that when students are asked to provide reasoning or a justification for their solution, a successful response will include the following:

- A logical sequence of steps
- An argument that explains why those steps are appropriate
- An accurate interpretation of the solution (with units) in the context of the situation, if appropriate

In order to help students develop these communication skills, teachers can:

- Have students practice explaining their solutions orally to a small group or to the class.
- Present an incomplete argument or explanation and have students supplement it for greater clarity or completeness.
- Provide sentence starters, template guides, and communication tips to help scaffold the writing process.

The table that follows provides examples of questions and instructional strategies for teaching students to communicate and apply notation correctly throughout the course.

Mathematical Practice 4: *Communication and Notation*

Skills	Key Questions	Sample Activities	Sample Instructional Strategies
4.A: *Use precise mathematical language.*	▪ Could this be more specific? ▪ Does the solution clarify which graph, function, or derivative I'm referring to?	Have students find and correct the errors in sample responses justifying the behavior of f based on the graph of f' or f'', specifically identifying vague language like "it" or "the function."	▪ Error Analysis ▪ Match Mine ▪ Model Questions ▪ Password-Style Games ▪ Round Table
4.B: *Use appropriate units of measure.*	▪ Are these units appropriate for this solution? ▪ What do these units mean in the context of the given problem?	Have students explore negative rates of change with a scenario problem, for example, the velocity of a basketball tossed directly upward at a velocity of 16 ft/s is decreasing at a constant rate of $16\,\frac{\text{ft/sec}}{\text{sec}}$, so the velocity of the ball in ft/s as a function of time in seconds is given by $(t) = 16 - 16t$. Have students graph the velocity function as a function of time over the interval $t = 0$ to $t = 2$ seconds, find the area of the region bounded by the velocity function and the time axis between $t = 0$ and $t = 1$, and then interpret the meaning of the area, including units.	▪ Error Analysis ▪ Marking the Text

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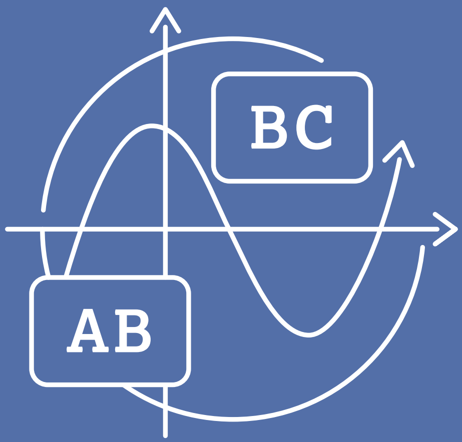
Mathematical Practice 4: *Communication and Notation (cont'd)*

Skills	Key Questions	Sample Activities	Sample Instructional Strategies
4.C: *Use appropriate mathematical symbols and notation (e.g., Represent a derivative using $f'(x)$, y', $\frac{dy}{dx}$).*	▪ How do we read this notation?	Have students match notational expressions to their verbal descriptions.	▪ Error Analysis ▪ Match Mine ▪ Notation Read Aloud
4.D: *Use appropriate graphing techniques.*	▪ What would a graph of this look like? ▪ Why is ___ a more appropriate representation than ___?	Have students match differential equations to their respective slope fields.	▪ Create Representations ▪ Graph and Switch
4.E: *Apply appropriate rounding procedures.*	▪ Have I saved the values for each step into my calculator before rounding? ▪ Has this question specified a particular number of decimal places to round to? ▪ Does my solution contain at least three decimal places?	Have students examine solutions for finding the volume of a solid of revolution using the washer method to determine whether appropriate rounding procedures were used.	▪ Collaborative Poster ▪ Identify a Subtask ▪ Scavenger Hunt

AP CALCULUS AB AND BC

Exam Information


BC
AB

Exam Overview

The AP Calculus AB and BC Exams assess student understanding of the mathematical practices and learning objectives outlined in the course framework. The exams are both 3 hours and 15 minute long and include 45 multiple-choice questions and 6 free-response questions. The details of the exams, including exam weighting, timing, and calculator requirements, can be found below:

Section	Question Type	Number of Questions	Exam Weighting	Timing
I	**Multiple-choice questions**			
	Part A: Graphing calculator not permitted	30	33.3%	60 minutes
	Part B: Graphing calculator required	15	16.7%	45 minutes
II	**Free-response questions**			
	Part A: Graphing calculator required	2	16.7%	30 minutes
	Part B: Graphing calculator not permitted	4	33.3%	60 minutes

The exams assess content from the three big ideas of the course.
Big Idea 1: Change
Big Idea 2: Limits
Big Idea 3: Analysis of Functions

The AP Exams also assess each of the units of the course—eight units for AP Calculus AB and 10 for AP Calculus BC—with the following exam weighting on the multiple-choice section:

Exam Weighting for the Multiple-Choice Section of the AP Exam

	Exam Weighting	
Unit	**AB**	**BC**
Unit 1: Limits and Continuity	**10–12%**	**4–7%**
Unit 2: Differentiation: Definition and Basic Derivative Rules	**10–12%**	**4–7%**
Unit 3: Differentiation: Composite, Implicit, and Inverse Functions	**9–13%**	**4–7%**
Unit 4: Contextual Applications of Differentiation	**10–15%**	**6–9%**
Unit 5: Applying Derivatives to Analyze Functions	**15–18%**	**8–11%**
Unit 6: Integration and Accumulation of Change	**17–20%**	**17–20%**
Unit 7: Differential Equations	**6–12%**	**6–9%**
Unit 8: Applications of Integration	**10–15%**	**6–9%**
Unit 9: Parametric Equations, Polar Coordinates, and Vector-Valued Functions BC ONLY		**11–12%**
Unit 10: Infinite Sequences and Series BC ONLY		**17–18%**

How Student Learning Is Assessed on the AP Exam

Section I: Multiple-Choice

The first section of the AP Calculus AB and BC Exams includes 45 multiple-choice questions. Students are permitted to use a calculator for the final 15 questions (Part B). Both the AB and BC Exams include algebraic, exponential, logarithmic, trigonometric, and general types of functions. Both also include analytical, graphical, tabular, and verbal types of representations.

Mathematical Practices 1, 2, and 3 are assessed in in the multiple-choice section with the following exam weighting (Practice 4 is not assessed):

Exam Weighting for the Multiple-Choice Section of the AP Exam

Mathematical Practice	Exam Weighting
Practice 1: Implementing Mathematical Processes	**53–66%**
Practice 2: Connecting Representations	**18–28%**
Practice 3: Justification	**11–18%**

Section II: Free-Response

The six free-response questions on the AP Calculus AB and BC Exams include a variety of content topics across the units of of the course. The AB and BC Exams include three common free-response questions that assess content from the domain of the AB Calculus course. Both AP Exams include various types of functions and function representations and a roughly equal mix of procedural and conceptual tasks. They both also include at least two questions that incorporate a real-world context or scenario into the question.

All four mathematical practices are assessed in the free-response section with the following exam weighting:

Exam Weighting for the Free-Response Section of the AP Exam

	Exam Weighting	
Mathematical Practice	**AB**	**BC**
Practice 1: Implementing Mathematical Processes	**37–55%**	**37–59%**
Practice 2: Connecting Representations	**9–16%**	**9–16%**
Practice 3: Justification	**37–55%**	**37–59%**
Practice 4: Communication and Notation	**13–24%**	**9–20%**

Task Verbs Used in Free-Response Questions

The following task verbs are commonly used in the free-response questions:

- **Approximate:** Use rounded decimal values or other estimates in calculations, which require writing an expression to show work.
- **Calculate/Write an expression:** Write an appropriate expression or equation to answer a question. Unless otherwise directed, calculations also require evaluating an expression or solving an equation, but the expression or equation must also be presented to show work. “Calculate” tasks might also be formulated as “How many?” or “What is the value?”
- **Determine:** Apply an appropriate definition, theorem, or test to identify values, intervals, or solutions whose existence or uniqueness can be established. “Determine” tasks may also be phrased as “Find.”
- **Estimate:** Use models or representations to find approximate values for functions.
- **Evaluate:** Apply mathematical processes, including the use of appropriate rounding procedures, to find the value of an expression at a given point or over a given interval.
- **Explain:** Use appropriate definitions or theorems to provide reasons or rationales for solutions and conclusions. “Explain” tasks may also be phrased as “Give a reason for…”
- **Identify/Indicate:** Indicate or provide information about a specified topic, without elaboration or explanation.
- **Interpret:** Describe the connection between a mathematical expression or solution and its meaning within the realistic context of a problem, often including consideration of units.
- **Interpret (when given a representation):** Identify mathematical information represented graphically, numerically, analytically, and/or verbally, with and without technology.
- **Justify:** Identify a logical sequence of mathematical definitions, theorems, or tests to support an argument or conclusion, explain why these apply, and then apply them.
- **Represent:** Use appropriate graphs, symbols, words, and/or tables of numerical values to describe mathematical concepts, characteristics, and/or relationships.
- **Verify:** Confirm that the conditions of a mathematical definition, theorem, or test are met in order to explain why it applies in a given situation. Alternately, confirm that solutions are accurate and appropriate.

Sample AP Calculus AB and BC Exam Questions

The sample exam questions that follow illustrate the relationship between the course framework and the AP Calculus AB and BC Exams and serve as examples of the types of questions that appear on the exams. After the sample questions is a table that shows which skill, learning objective(s), and unit each question relates to. The table also provides the answers to the multiple-choice questions.

Section I: Multiple-Choice

PART A (AB OR BC)

Graphing calculators are not permitted on this part of the exam.

1. $\lim_{x \to 0} \frac{1-\cos^2(2x)}{(2x)^2} =$

   (A) 0

   (B) $\frac{1}{4}$

   (C) $\frac{1}{2}$

   (D) 1

$$f(x) = \begin{cases} \frac{2}{x} & \text{for } x < -1 \\ x^2 - 3 & \text{for } -1 \le x \le 2 \\ 4x - 3 & \text{for } x > 2 \end{cases}$$

2. Let f be the function defined above. At what values of x, if any, is f not differentiable?

   (A) $x = -1$ only

   (B) $x = 2$ only

   (C) $x = -1$ and $x = -2$

   (D) f is differentiable for all values of x.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	2	−4	−5	3
2	−3	1	8	4

3. The table above gives values of the differentiable functions f and g and their derivatives at selected values of x. If h is the function defined by $h(x) = f(x)g(x) + 2g(x)$, then $h'(1) =$

(A) 32

(B) 30

(C) −6

(D) −16

4. If $x^3 - 2xy + 3y^2 = 7$, then $\dfrac{dy}{dx} =$

(A) $\dfrac{3x^2 + 4y}{2x}$

(B) $\dfrac{3x^2 - 2y}{2x - 6y}$

(C) $\dfrac{3x^2}{2x - 6y}$

(D) $\dfrac{3x^2}{2 - 6y}$

5. The radius of a right circular cylinder is increasing at a rate of 2 units per second. The height of the cylinder is decreasing at a rate of 5 units per second. Which of the following expressions gives the rate at which the volume of the cylinder is changing with respect to time in terms of the radius r and height h of the cylinder?
(The volume V of a cylinder with radius r and height h is $V = \pi r^2 h$.)

(A) $-20\pi r$

(B) $-2\pi rh$

(C) $4\pi rh - 5\pi r^2$

(D) $4\pi rh + 5\pi r^2$

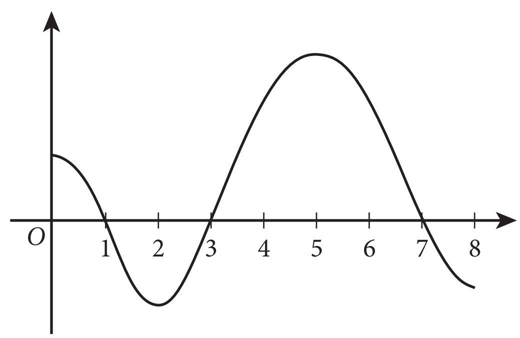
6. Which of the following is equivalent to the definite integral $\int_2^6 \sqrt{x}\, dx$?

(A) $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{4}{n}\sqrt{\frac{4k}{n}}$

(B) $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{6}{n}\sqrt{\frac{6k}{n}}$

(C) $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{4}{n}\sqrt{2+\frac{4k}{n}}$

(D) $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{6}{n}\sqrt{2+\frac{6k}{n}}$



Graph of g

7. The figure above shows the graph of the continuous function g on the interval $[0, 8]$. Let h be the function defined by $h(x) = \int_3^x g(t)\,dt$. On what intervals is h increasing?

(A) $[2, 5]$only

(B) $[1, 7]$

(C) $[0, 1]$ and $[3, 7]$

(D) $[1, 3]$ and $[7, 8]$

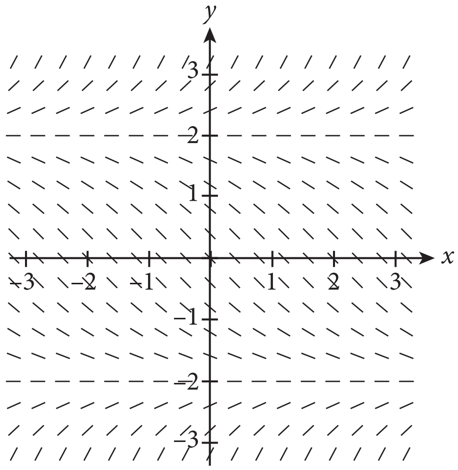
8. $\int \frac{x}{\sqrt{1-9x^2}}\,dx =$

(A) $-\frac{1}{9}\sqrt{1-9x^2} + C$

(B) $-\frac{1}{18}\ln \sqrt{1-9x^2} + C$

(C) $\frac{1}{3}\arcsin(3x) + C$

(D) $\frac{x}{3}\arcsin(3x) + C$



9. Shown above is a slope field for which of the following differential equations?

(A) $\dfrac{dy}{dx} = \dfrac{y-2}{2}$

(B) $\dfrac{dy}{dx} = \dfrac{y^2-4}{4}$

(C) $\dfrac{dy}{dx} = \dfrac{x-2}{2}$

(D) $\dfrac{dy}{dx} = \dfrac{x^2-4}{4}$

10. Let R be the region bounded by the graph of $x = e^y$, the vertical line $x = 10$, and the horizontal lines $y = 1$ and $y = 2$. Which of the following gives the area of R?

(A) $\int_1^2 e^y\, dy$

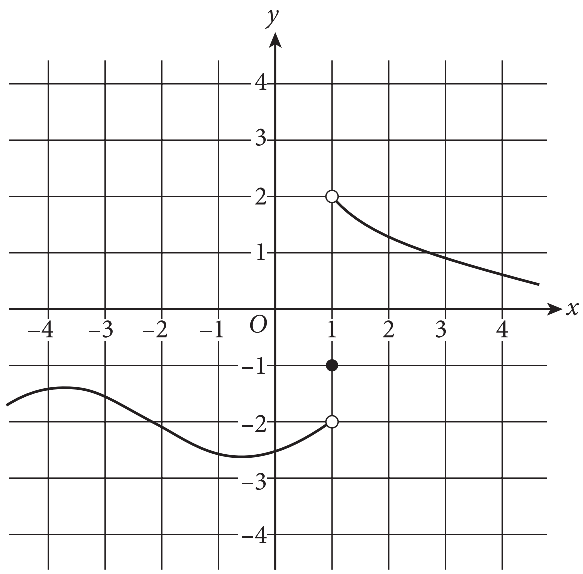
(B) $\int_e^{e^2} \ln x\, dx$

(C) $\int_1^2 (10 - e^y)dy$

(D) $\int_e^{10} (\ln x - 1)dx$

PART B (AB OR BC)

A graphing calculator is required on this part of the exam.



Graph of f

11. The graph of the function f is shown in the figure above. The value of $\lim\limits_{x \to 1^+} f(x)$ is

    (A) -2

    (B) -1

    (C) 2

    (D) nonexistent

12. The velocity of a particle moving along a straight line is given by $v(t) = 1.3t\ln(0.2t + 0.4)$ for time $t \geq 0$. What is the acceleration of the particle at time $t = 1.2$?

    (A) -0.580

    (B) -0.548

    (C) -0.093

    (D) 0.660

x	-1	0	2	4	5
$f'(x)$	11	9	8	5	2

13. Let f be a twice-differentiable function. Values of f', the derivative of f, at selected values of x are given in the table above. Which of the following statements must be true?

(A) f is increasing for $-1 \le x \le 5$.

(B) The graph of f is concave down for $-1 < x < 5$.

(C) There exists c, where $-1 < c < 5$, such that $f'(c) = -\frac{3}{2}$.

(D) There exists c, where $-1 < c < 5$, such that $f''(c) = -\frac{3}{2}$.

14. Let f be the function with derivative defined by $f'(x) = 2 + (2x - 8)\sin(x + 3)$. How many points of inflection does the graph of f have on the interval $0 < x < 9$?

(A) One

(B) Two

(C) Three

(D) Four

15. Honey is poured through a funnel at a rate of $r(t) = 4e^{-0.35t}$ ounces per minute, where t is measured in minutes. How many ounces of honey are poured through the funnel from time $t = 0$ to time $t = 3$?

(A) 0.910

(B) 1.400

(C) 2.600

(D) 7.429

PART A (BC ONLY)

Graphing calculators are not permitted on this part of the exam.

x	2	5
$f(x)$	4	7
$f'(x)$	2	3

16. The table above gives values of the differentiable function f and its derivative f' at selected values of x.

If $\int_2^5 f(x)dx = 14$, what is the value of $\int_2^5 x \cdot f'(x)dx$?

(A) 13

(B) 27

(C) $\frac{63}{2}$

(D) 41

17. The number of fish in a lake is modeled by the function F that satisfies the logistic differential equation $\frac{dF}{dt} = 0.04F\left(1 - \frac{F}{5000}\right)$, where t is the time in months and $F(0) = 2000$. What is $\lim_{t \to \infty} F(t)$?

(A) 10,000

(B) 5000

(C) 2500

(D) 2000

18. A curve is defined by the parametric equations $x(t) = t^2 + 3$ and $y(t) = \sin(t^2)$. Which of the following is an expression for $\frac{d^2y}{dx^2}$ in terms of t?

(A) $-\sin(t^2)$

(B) $-2t\sin(t^2)$

(C) $\cos(t^2) - 2t^2\sin(t^2)$

(D) $2\cos(t^2) - 4t^2\sin(t^2)$

19. Which of the following series is conditionally convergent?

(A) $\sum_{k=1}^{\infty} (-1)^k \frac{5}{k^3+1}$

(B) $\sum_{k=1}^{\infty} (-1)^k \frac{5}{k+1}$

(C) $\sum_{k=1}^{\infty} (-1)^k \frac{5k}{k+1}$

(D) $\sum_{k=1}^{\infty} (-1)^k \frac{5k^2}{k+1}$

20. Let f be the function defined by $f(x) = e^{2x}$. Which of the following is the Maclaurin series for f', the derivative of f?

(A) $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$

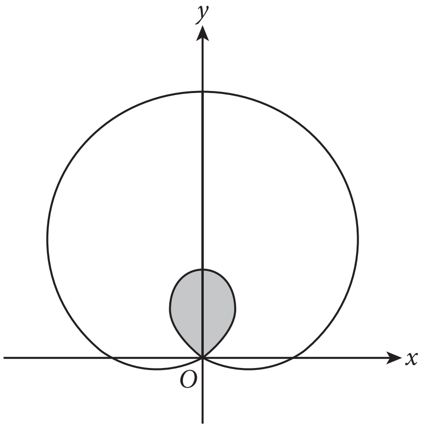
(B) $2 + 2x + \frac{2x^2}{2!} + \frac{2x^3}{3!} + \cdots + \frac{2x^n}{n!} + \cdots$

(C) $1 + 2x + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \cdots + \frac{(2x)^n}{n!} + \cdots$

(D) $2 + 2(2x) + \frac{2(2x)^2}{2!} + \frac{2(2x)^3}{3!} + \cdots + \frac{2(2x)^n}{n!} + \cdots$

PART B (BC ONLY)

A graphing calculator is required on this part of the exam.



21. The figure above shows the graph of the polar curve $r = 2 + 4\sin\theta$. What is the area of the shaded region?

    (A) 2.174

    (B) 2.739

    (C) 13.660

    (D) 37.699

22. The function f has derivatives of all orders for all real numbers. It is known that $\left|f^{(4)}(x)\right| \le \frac{12}{5}$ and $\left|f^{(5)}(x)\right| \le \frac{3}{2}$ for $0 \le x \le 2$. Let $P_4(x)$ be the fourth-degree Taylor polynomial for f about $x = 0$. The Taylor series for f about $x = 0$ converges at $x = 2$. Of the following, which is the smallest value of k for which the Lagrange error bound guarantees that $\left|f(2) - P_4(2)\right| \le k$?

    (A) $\frac{2^5}{5!} \cdot \frac{3}{2}$

    (B) $\frac{2^5}{5!} \cdot \frac{12}{5}$

    (C) $\frac{2^4}{4!} \cdot \frac{3}{2}$

    (D) $\frac{2^4}{4!} \cdot \frac{12}{5}$

Section II: Free-Response

The following are examples of the kinds of free-response questions found on the exam.

PART A (AB OR BC)

A graphing calculator is required on this part of the exam.

t (hours)	0	2	4	6	8	10	12
$R(t)$ (vehicles per hour)	2935	3653	3442	3010	3604	1986	2201

1. On a certain weekday, the rate at which vehicles cross a bridge is modeled by the differentiable function R for $0 \le t \le 12$, where $R(t)$ is measured in vehicles per hour and t is the number of hours since 7:00 a.m. ($t = 0$). Values of $R(t)$ for selected values of t are given in the table above.

   (a) Use the data in the table to approximate $R'(5)$. Show the computations that lead to your answer. Using correct units, explain the meaning of $R'(5)$ in the context of the problem.

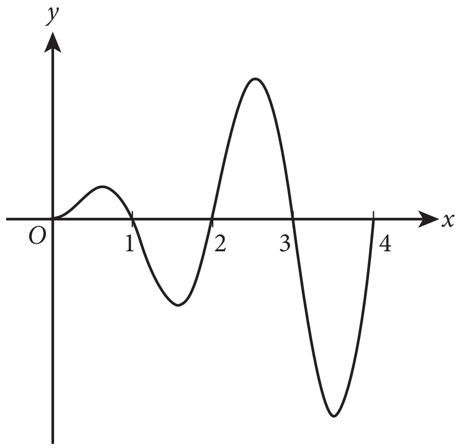
   (b) Use a midpoint sum with three subintervals of equal length indicated by the data in the table to approximate the value of $\int_0^{12} R(t)\,dt$. Indicate units of measure.

   (c) On a certain weekend day, the rate at which vehicles cross the bridge is modeled by the function H defined by $H(t) = -t^3 - 3t^2 + 288t + 1300$ for $0 \le t \le 17$, where $H(t)$ is measured in vehicles per hour and t is the number of hours since 7:00 a.m. ($t = 0$). According to this model, what is the average number of vehicles crossing the bridge per hour on the weekend day for $0 \le t \le 12$?

   (d) For $12 < t < 17$, $L(t)$, the local linear approximation to the function H given in part (c) at $t = 12$, is a better model for the rate at which vehicles cross the bridge on the weekend day. Use $L(t)$ to find the time t, for $12 < t < 17$, at which the rate of vehicles crossing the bridge is 2000 vehicles per hour. Show the work that leads to your answer.

PART B (AB OR BC)

Graphing calculators are not permitted on this part of the exam.



Graph of f'

2. The figure above shows the graph of f', the derivative of a twice-differentiable function f, on the closed interval $[0, 4]$. The areas of the regions bounded by the graph of f' and the x-axis on the intervals $[0, 1]$, $[1, 2]$, $[2, 3]$, and $[3, 4]$ are 2, 6, 10, and 14, respectively. The graph of f' has horizontal tangents at $x = 0.6$, $x = 1.6$, $x = 2.5$, and $x = 3.5$. It is known that $f(2) = 5$.
   (a) On what open intervals contained in $(0, 4)$ is the graph of f both decreasing and concave down? Give a reason for your answer.
   (b) Find the absolute minimum value of f on the interval $[0, 4]$. Justify your answer.
   (c) Evaluate $\int_0^4 f(x)f'(x)dx$.
   (d) The function g is defined by $g(x) = x^3 f(x)$. Find $g'(2)$. Show the work that leads to your answer.

PART A (BC ONLY)

A graphing calculator is required on this part of the exam.

3. For $0 \le t \le 5$, a particle is moving along a curve so that its position at time t is $(x(t), y(t))$. At time $t = 1$, the particle is at position $(2, -7)$. It is known that $\frac{dx}{dt} = \sin\left(\frac{t}{t+3}\right)$ and $\frac{dy}{dt} = e^{\cos t}$.
   (a) Write an equation for the line tangent to the curve at the point $(2, -7)$.
   (b) Find the y-coordinate of the position of the particle at time $t = 4$.
   (c) Find the total distance traveled by the particle from time $t = 1$ to time $t = 4$.
   (d) Find the time at which the speed of the particle is 2.5. Find the acceleration vector of the particle at this time.

PART B (BC ONLY)

Graphing calculators are not permitted on this part of the exam.

4. The Maclaurin series for the function *f* is given by

$$f(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k^2} = x - \frac{x^2}{4} + \frac{x^3}{9} - \cdots \text{ on its interval of convergence.}$$

(a) Use the ratio test to determine the interval of convergence of the Maclaurin series for *f*. Show the work that leads to your answer.

(b) The Maclaurin series for *f* evaluated at $x = \frac{1}{4}$ is an alternating series whose terms decrease in absolute value to 0. The approximation for $f\left(\frac{1}{4}\right)$ using the first two nonzero terms of this series is $\frac{15}{64}$. Show that this approximation differs from $f\left(\frac{1}{4}\right)$ by less than $\frac{1}{500}$.

(c) Let *h* be the function defined by $h(x) = \int_0^x f(t)\,dt$. Write the first three nonzero terms and the general term of the Maclaurin series for *h*.

Answer Key and Question Alignment to Course Framework

Multiple-Choice Question	Answer	Skill	Learning Objective	Unit
1	D	1.E	LIM-1.E	1
2	B	3.D	FUN-2.A	2
3	A	1.E	FUN-3.B	2
4	B	1.E	FUN-3.D	3
5	C	1.E	CHA-3.D	4
6	C	2.C	LIM-5.C	6
7	C	2.D	FUN-5.A	6
8	A	1.E	FUN-6.D	6
9	B	2.C	FUN-7.C	7
10	C	1.D	CHA-5.A	8
11	C	2.B	LIM-1.C	1
12	C	1.E	CHA-3.B	4
13	D	3.D	FUN-1.B	5
14	D	2.D	FUN-4.A	5
15	D	3.D	CHA-4.D	8
16	A	1.E	FUN-6.E	6
17	B	3.F	FUN-7.H	7
18	A	1.E	CHA-3.G	9
19	B	3.D	LIM-7.A	10
20	D	3.D	LIM-8.G	10
21	A	3.D	CHA-5.D	9
22	A	1.F	LIM-8.C	10

Free-Response Question	Skills	Learning Objective	Unit
1	1.D, 1.E, 2.B, 3.F	CHA-2.D, CHA-3.A, CHA-3.C, CHA-3.F, CHA-4.B, LIM-5.A	2, 4, 6, 8
2	1.C, 1.E, 2.B, 2.E, 3.B, 3.E	FUN-3.B, FUN-4.A, FUN-5.A, FUN-6.D	2, 5, 6
3	1.C, 1.D, 1.E, 2.B	CHA-3.G, FUN-8.B	9
4	1.D, 1.E, 3.B, 3.D, 3.E	LIM-7.A, LIM-7.B, LIM-8.D, LIM-8.G	10

The scoring information for the questions within this course and exam description, along with further exam resources, can be found on the **AP Calculus AB Exam Page** and the **AP Calculus BC Exam Page** on AP Central.

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